Homework 3

due October 22, 2003

Reminder: There will be a midterm on Monday October 20! That's why this homework is a little bit shorter than usual.

1. Try to answer the following question that Paul asked in class. Justify your answer:
Let $a_1, a_2, \ldots, a_n$ be integers not all equal to zero. Is it true that the greatest common divisor of these integers $(a_1, \ldots, a_n)$ is the least positive integer of the form $m_1a_1+m_2a_2+\cdots+m_na_n$ where $m_1, \ldots, m_n \in \mathbb{Z}$? (see also Rosen 3.2 #20, pg. 85)

2. Rosen 3.4 #8, pg. 104
Show that every positive integer can be written as the product of possibly a square and a square-free integer. A *square-free* integer is an integer that is not divisible by any perfect squares other than 1.

3. Rosen 3.4 #19 and #22, pg. 105
Let $\alpha = a + b\sqrt{-5}$ where $a, b \in \mathbb{Z}$. Define the *norm* of α, denoted $N(\alpha)$, as $N(\alpha) = a^2 + 5b^2$.
(a) Show that if $\alpha = a+b\sqrt{-5}$ and $\beta = c+d\sqrt{-5}$, where $a, b, c, d \in \mathbb{Z}$, then $N(\alpha\beta) = N(\alpha)N(\beta)$.
(b) Show that the numbers $1+\sqrt{-5}$ and $1-\sqrt{-5}$ are prime numbers, that is, there are no numbers $\alpha = a+b\sqrt{-5}$ and $\beta = c+d\sqrt{-5}$ different from ± 1 such that $1 \pm \sqrt{-5} = \alpha\beta$.
(Hint: Use part (a)).

4. Rosen 3.4 #45, pg. 107
Show that $\sqrt{2}+\sqrt{3}$ is irrational.