Practice Problems

Here are some review questions. Make sure that you can state all theorems/definitions precisely with all assumptions:

(1) What is the well-ordering principle?
(2) What is the principle of induction?
(3) Define the Fibonacci numbers.
(4) Write down the division algorithm.
(5) What is the prime number theorem?
(6) Is there a bound on the number of consecutive composite integers?
(7) Write down Dirichlet's theorem on primes in arithmetic progressions.
(8) What is the twin prime conjecture?
(9) What is Goldbach's conjecture?
(10) Define the greatest common divisor of two integers a and b.
(11) What is the relation between (a, b) and integers of the form $ma + nb$ where $m, n \in \mathbb{Z}$?
(12) State the Euclidean algorithm.
(13) State the fundamental theorem of arithmetic.
(14) Define the least common multiple of two integers a and b. What is the relation between $[a, b]$ and (a, b)?
(15) How does Fermat's factorization method work?
(16) What do you know about solutions to linear Diophantine equations of the form $ax + by = c$?
(17) What is a perfect number?
(18) What is a Mersenne number?
(19) What is the relation between perfect numbers and Mersenne numbers?
(20) What is a complete set of residues mod m?
(21) Explain how to calculate $3^{10} \pmod{11}$ using modular exponentiation.
(22) What do you know about solutions to linear congruences of the form $ax \equiv b \pmod{m}$?
(23) When does the inverse of a modulo m exist?
(24) State the Chinese remainder theorem.
(25) State Wilson's theorem.
(26) State Fermat's little theorem.
(27) State Euler's theorem.

(28) Define a reduced residue system modulo n.
(29) What is a pseudoprime to base a?
(30) Write down the definition of a Carmichael number.
(31) What is the Korselt's criterion?
(32) State the Miller primality test.
(33) Define the Euler Φ function. Write down a formula for $\Phi(n)$.
(34) What is $\sum_{d|n} \Phi(d)$?
(35) Define the functions $\tau(n)$ and $\sigma(n)$ and write down their properties and formulas for them.
(36) What is a summatory function?
(37) What is a multiplicative function?
(38) Define the Möbius function.
(39) What is $\sum_{d|n} \mu(d)$?
(40) Write down the Möbius inversion formula.
(41) Explain the RSA encryption method.
(42) What is a super-increasing sequence?
(43) State the knapsack problem.

Next there are some practice problems. The solutions will be discussed in class 12/3 and 12/5:

1. Prove that the difference of the square of two consecutive Fibonacci numbers is equal to the product of two Fibonacci numbers.

2. Let $g_1 = 1$, $g_2 = 4$ and $g_n = g_{n-1} + g_{n-2}$ for $n > 2$. Find and prove a formula for $\sum_{i=1}^{n} g_i$.

3. Show that $(13n + 5, 8n + 3) = 1$ for all positive integers n.

4. A postal clerk has only 14- and 21-cent stamps. What combinations can be used to make up \$3.50 worth of stamps?

5. Prove that there exists a string of 50 consecutive integers each of which is divisible by a perfect cube.

6. Determine the last two digits of $3^{3^{100}}$. [*Hint:* $\Phi(100) = 40$ and $\Phi(40) = 16$.]

7. Find solutions to $9x \equiv 2 \pmod{49}$ using Euler's theorem.

8. Prove that there are infinitely many solutions to the equation $\tau(n) = 2$.

9. Prove that $\sum_{d|n} \mu(d)\tau(n/d) = 1$.

10. In the RSA encryption system choose $n = 65$. Find the decryption key d for $e = 5$ and for $e = 7$. Encrypt the message $P = 03$ with $e = 5$.