Homework Set One: Complex Numbers

Directions: Submit your solutions to the Calculational Exercises and the Proof-Writing Exercises **separately** at the **beginning** of lecture on **Friday, October 5, 2007**. The two problems sets will be graded by different persons.

Calculational Exercises

Submit solutions to Exercises 1(b), 2(a, b, c), 3(b), 4(a, b), and 5(a, b).

1. Solve the following systems of linear equations and characterize their solution set. (I.e., determine whether there is a unique solution, no solution, etc.) Also, write each system of linear equations as a single function $f : \mathbb{R}^n \to \mathbb{R}^m$ for appropriate choices of $m, n \in \mathbb{Z}_+$.

   (a) System of 3 equations in the unknowns x, y, z, w:

   $$\left.\begin{aligned} x + 2y - 2z + 3w &= 2 \\ 2x + 4y - 3z + 4w &= 5 \\ 5x + 10y - 8z + 11w &= 12 \end{aligned}\right\}.$$

   (b) System of 4 equations in the unknowns x, y, z:

   $$\left.\begin{aligned} x + 2y - 3z &= 4 \\ x + 3y + z &= 11 \\ 2x + 5y - 4z &= 13 \\ 2x + 6y + 2z &= 22 \end{aligned}\right\}.$$

   (c) System of 3 equations in the unknowns x, y, z:

   $$\left.\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}\right\}.$$

2. Express the following complex numbers in the form $x + yi$ for $x, y \in \mathbb{R}$:

   (a) $(2 + 3i) + (4 + i)$

(b) $(2+3i)^2(4+i)$

(c) $\dfrac{2+3i}{4+i}$

(d) $\dfrac{1}{i}+\dfrac{3}{1+i}$

(e) $(-i)^{-1}$

(f) $(-1+i\sqrt{3})^3$

3. Compute the real and imaginary parts of the following expressions, where z is the complex number $x+yi$ and $x, y \in \mathbb{R}$:

   (a) $\dfrac{1}{z^2}$

   (b) $\dfrac{1}{3z+2}$

   (c) $\dfrac{z+1}{2z-5}$

   (d) z^3

4. Solve the following equations for z a complex number:

   (a) $z^5 - 2 = 0$

   (b) $z^4 + i = 0$

   (c) $z^6 + 8 = 0$

   (d) $z^3 - 4i = 0$

5. Compute the real and imaginary parts:

   (a) e^{2+i}

   (b) $\sin(1+i)$

   (c) e^{3-i}

   (d) $\cos(2+3i)$

Proof-Writing Exercises

Submit solutions to Exercises 1 and 2.

1. Let a, b, c, and d be real numbers, and consider the system of equations given by

$$ax_1 + bx_2 = 0, \tag{1}$$
$$cx_1 + dx_2 = 0. \tag{2}$$

Note that $x_1 = x_2 = 0$ is a solution for any choice of a, b, c, and d. Prove that if $ad - bc \neq 0$, then $x_1 = x_2 = 0$ is the only solution.

2. Let $a \in \mathbb{R}$ and $z, w \in \mathbb{C}$. Prove that

   (a) $\mathrm{Re}(az) = a\mathrm{Re}(z)$ and $\mathrm{Im}(az) = a\mathrm{Im}(z)$.

   (b) $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$ and $\mathrm{Im}(z + w) = \mathrm{Im}(z) + \mathrm{Im}(w)$.