Homework Set Six: Eigenvalues

Directions: Submit your solutions to the Calculational Exercises and the Proof-Writing Exercises **separately** at the **beginning** of lecture on **Friday, November 9, 2007**. The two problems sets will be graded by different persons.

Calculational Exercises

Do Problem 1 and 2(a),(b).

1. Let $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^2)$ be defined by

$$T(u, v) = (v, u)$$

for every $u, v \in \mathbb{F}$. Compute the eigenvalues and associated eigenvectors for T.

2. Find eigenvalues and associated eigenvectors for the linear operators on $\mathbb{F}^2$ defined by each given 2×2 matrix.

(a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hint: Use the fact that, given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}^{2\times 2}$, $\lambda \in \mathbb{F}$ is an eigenvalue for A if and only if $(a - \lambda)(d - \lambda) - bc = 0$.

Proof-Writing Exercises

1. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $T \in \mathcal{L}(V, V)$, and let $U_1, \ldots, U_m$ be subspaces of V that are invariant under T. Prove that $U_1 + \cdots + U_m$ must then also be an invariant subspace of V under T.

2. Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose that the linear operator $P \in \mathcal{L}(V)$ has the property that $P^2 = P$. Prove that $V = \text{null}(P) \oplus \text{range}(P)$.