Homework Set Seven: Permutations and more on Eigenvalues

Directions: Submit your solutions to the Calculational Exercises and the Proof-Writing Exercises **separately** at the **beginning** of lecture on **Friday, November 16, 2007**. The two problems sets will be graded by different persons.

Calculational Exercises

1. Let $A \in \mathbb{C}^{3\times 3}$ be given by

$$A = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & -1 \end{bmatrix}.$$

   (a) Calculate $\det(A)$.

   (b) Find $\det(A^4)$.

2. (a) For each permutation $\pi \in \mathcal{S}_3$, compute the number of inversions in π, and classify π as being either an even or an odd permutation.

   (b) Use your result from Part (a) to construct a formula for the determinant of a 3×3 matrix.

Proof-Writing Exercises

1. (a) Let $a, b, c, d \in \mathbb{F}$ and consider the system of equations given by

$$ax_1 + bx_2 = 0 \tag{1}$$
$$cx_1 + dx_2 = 0. \tag{2}$$

   Note that $x_1 = x_2 = 0$ is a solution for any choice of a, b, c, and d. Prove that this system of equations has a non-trivial solution if and only if $ad - bc = 0$.

   (b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}^{2\times 2}$, and recall that we can define a linear operator $T \in \mathcal{L}(\mathbb{F}^2)$ on $\mathbb{F}^2$ by setting $T(v) = Av$ for each $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{F}^2$.

Show that the eigenvalues for T are exactly the $\lambda \in \mathbb{F}$ for which $p(\lambda) = 0$, where $p(z) = (a - z)(d - z) - bc$.

Hint: Write the eigenvalue equation $Av = \lambda v$ as $(A - \lambda I)v = 0$ and use the first part.

2. Prove or give a counterexample: For any $n \geq 1$ and $A, B \in \mathbb{R}^{n\times n}$, one has

$$\det(A + B) = \det(A) + \det(B).$$