Homework Set One: Complex Numbers

Directions: Submit your solutions to the Calculational Exercises and the Proof-Writing Exercises at the **beginning** of lecture on **Friday, October 2, 2009**.

Calculational Exercises

Submit solutions to Exercises 1(b), 2(a, b, c), 3(b), 4(a, b), and 5(a, b).

1. Solve the following systems of linear equations and characterize their solution sets. (I.e., determine whether there is a unique solution, no solution, etc.) Also, write each system of linear equations as a single function $f : \mathbb{R}^n \to \mathbb{R}^m$ for appropriate choices of $m, n \in \mathbb{Z}_+$.

   (a) System of 3 equations in the unknowns x, y, z, w:

   $$\left.\begin{aligned} x + 2y - 2z + 3w &= 2 \\ 2x + 4y - 3z + 4w &= 5 \\ 5x + 10y - 8z + 11w &= 12 \end{aligned}\right\} .$$

   (b) System of 4 equations in the unknowns x, y, z:

   $$\left.\begin{aligned} x + 2y - 3z &= 4 \\ x + 3y + z &= 11 \\ 2x + 5y - 4z &= 13 \\ 2x + 6y + 2z &= 22 \end{aligned}\right\} .$$

   (c) System of 3 equations in the unknowns x, y, z:

   $$\left.\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}\right\} .$$

2. Express the following complex numbers in the form $x + yi$ for $x, y \in \mathbb{R}$:

   (a) $(2 + 3i) + (4 + i)$

   (b) $(2 + 3i)^2(4 + i)$

(c) $\dfrac{2+3i}{4+i}$

(d) $\dfrac{1}{i} + \dfrac{3}{1+i}$

(e) $(-i)^{-1}$

(f) $(-1+i\sqrt{3})^3$

3. Compute the real and imaginary parts of the following expressions, where z is the complex number $x + yi$ and $x, y \in \mathbb{R}$:

   (a) $\dfrac{1}{z^2}$

   (b) $\dfrac{1}{3z+2}$

   (c) $\dfrac{z+1}{2z-5}$

   (d) z^3

4. Solve the following equations for z a complex number:

   (a) $z^5 - 2 = 0$

   (b) $z^4 + i = 0$

   (c) $z^6 + 8 = 0$

   (d) $z^3 - 4i = 0$

5. Compute the real and imaginary parts:

   (a) e^{2+i}

   (b) $\sin(1+i)$

   (c) e^{3-i}

   (d) $\cos(2+3i)$

Proof-Writing Exercises

Submit solutions to Exercises 1 and 2.

1. Let $a \in \mathbb{R}$ and $z, w \in \mathbb{C}$. Prove that

   (a) $\mathrm{Re}(az) = a\mathrm{Re}(z)$ and $\mathrm{Im}(az) = a\mathrm{Im}(z)$.

(b) $\mathrm{Re}(z+w) = \mathrm{Re}(z) + \mathrm{Re}(w)$ and $\mathrm{Im}(z+w) = \mathrm{Im}(z) + \mathrm{Im}(w)$.

2. Let $z, w \in \mathbb{C}$ with $\overline{z}w \neq 1$ such that either $|z| = 1$ or $|w| = 1$. Prove that $\left|\dfrac{z-w}{1-\overline{z}w}\right| = 1$.