Homework 3
due October 24, 2014 in class

Read: Artin Ch. 2.5, 2.6

1. Artin 2.6.8 (pg. 72)
   Prove that the map $f\colon A \mapsto (A^t)^{-1}$ is an automorphism of $GL_n(\mathbb{R})$.

2. Prove that the kernel and image of a homomorphism are subgroups.

3. Find all subgroups of S_3, and determine which are normal.

4. Let $\varphi : G \to G'$ be a group homomorphism. Prove that $\varphi(x) = \varphi(y)$ if and only if $xy^{-1} \in \ker \varphi$.

5. (Artin 2.6.7 (pg. 72))

   (a) Let H be a subgroup of G, and let g be a fixed element of G. The *conjugate subgroup* gHg^{-1} is defined to be the set of all conjugates ghg^{-1}, with $h \in H$. Prove that gHg^{-1} is a subgroup of G.

   (b) Prove that a subgroup H of G is normal if and only if $gHg^{-1} = H$ for all $g \in G$.

6. Prove that the center of a group is a normal subgroup.

7. If $\varphi\colon G \to H$ is an isomorphism, prove that $|\varphi(x)| = |x|$ for all $x \in G$. Here $|x|$ denotes the order of the element x. Deduce that any two isomorphic groups have the same number of elements of order n for each $n \in \mathbb{Z}^+$. Is the result true if φ is only assumed to be a homomorphism?