Homework 5
due May 16, 2014 in class

(1) (Artin 14.2.4(b)) Let I be an ideal of a ring R. Prove or disprove: If R/I is a free R-module, then $I = 0$.
(2) (Artin 14.5.1) Let $R = \mathbb{Z}[\delta]$, where $\delta = \sqrt{-5}$. Determine a presentation matrix as R-module for the ideal $(2, 1 + \delta)$.
(3) Let A be the presentation matrix of a module V with respect to a set of generators $(v_1, \ldots, v_m)$. Let $(w_1, \ldots, w_r)$ be another set of elements of V, and write the elements in terms of the generators, say $w_i = \sum_j p_{ij} v_j$, where $p_{ij} \in R$. Let $P = (p_{ij})$. Prove that the block matrix
$$\begin{pmatrix} A & -P^t \\ 0 & I \end{pmatrix}$$
is a presentation matrix for V with respect to the set of generators $(v_1, \ldots, v_m; w_1, \ldots, w_r)$.
(4) Let R be a ring, and let V be an R-module, presented by a diagonal $m \times n$ matrix A: $V \cong R^m/AR^n$. Let $(v_1, \ldots, v_m)$ be the corresponding generators of V, and let d_i be the diagonal entries of A. Prove that V is isomorphic to a direct product of the modules $R/(d_i)$.
(5) (Artin 14.7.7) Let $R = \mathbb{Z}[i]$ and let V be the R-module generated by elements v_1 and v_2 with relations $(1+i)v_1 + (2-i)v_2 = 0$, $3v_1 + 5iv_2 = 0$. Write this module as a direct sum of cyclic modules.
(6) Let $W_1, \ldots, W_k$ be submodules of an R-module V such that $V = \sum_i W_i$. Assume that $W_1 \cap W_2 = 0$, $(W_1 + W_2) \cap W_3 = 0, \ldots, (W_1 + W_2 + \cdots + W_{k-1}) \cap W_k = 0$. Prove that V is the direct sum of the modules $W_1, \ldots, W_k$.
(7) Give a list of all abelian groups of order 180 (up to isomorphism).