Homework 2

due January 28, 2004

Question 1. Show that the complex derivative has the following properties (for problems (3)-(7) we assume that f and g are analytic on an appropriate open set):

(1) If $f'(z_0)$ exists, then f is continuous at z_0.

(2) $$\frac{d}{dz} z^n = n z^{n-1}$$

(3) $$\frac{d}{dz}(f+g) = \frac{df}{dz} + \frac{dg}{dz}$$

(4) $$\frac{d}{dz}(cf) = c\frac{df}{dz} \quad \text{for } c \in \mathbb{C}$$

(5) $$\frac{d}{dz}(f \cdot g) = \frac{df}{dz} g + f\frac{dg}{dz}$$

(6) $$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{\frac{df}{dz} g - f\frac{dg}{dz}}{g^2} \quad \text{for } g(z) \neq 0$$

(7) $$\frac{d}{dz} f(g(z)) = f'(g(z)) g'(z)$$

Question 2. Compute df/dz when it exists for

(i) $f(z) = 1/z$
(ii) $f(z) = x^2 + iy^2$
(iii) $f(z) = z\mathrm{Im}(z)$
(iv) $f(z) = \overline{z}$

Question 3. Study

$$f(z) = \begin{cases} \frac{x^3 y(y-ix)}{x^6+y^2} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

Show that even though $[f(z) - f(0)]/z \to 0$ as $z \to 0$ for any straight line through the origin, $f(z)$ is nevertheless not complex differentiable at $z = 0$.

Question 4. Use the Cauchy–Riemann relations to show that an analytic function that takes only real values in some neighborhood, must be constant there.

Question 5. Define the symbols $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ by

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right).$$

(i) Show that the Cauchy–Riemann equations are equivalent to $\partial f/\partial \overline{z} = 0$.
(ii) An antianalytic function is defined by the condition $\partial f/\partial z = 0$. Derive the Cauchy-Riemann relations for antianalytic functions.
(iii) Show that if f is analytic, then $f' = \partial f/\partial z$.
(iv) If $f(z) = z$, show that $\partial f/\partial z = 1$ and $\partial f/\partial \overline{z} = 0$.
(v) If $f(z) = \overline{z}$, show that $\partial f/\partial z = 0$ and $\partial f/\partial \overline{z} = 1$.
(vi) Show that the symbols $\partial/\partial z$ and $\partial/\partial \overline{z}$ obey the sum, product, and scalar multiple rules for derivatives.
(vii) Show that the expression

$$\sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm} z^n \overline{z}^m$$

with scalar $a_{nm} \in \mathbb{C}$ is an analytic function if and only if $a_{nm} = 0$ whenever $m \neq 0$.

Question 6. Let us study the Cauchy–Riemann relations in polar coordinates. In what follows, assume that $f(z)$ is analytic and that $z = r\exp(i\theta)$.

(i) Using that df/dz can be computed for any direction of approach $dz \to 0$, show that

$$\frac{df}{dz} = e^{-i\theta}\frac{\partial f}{\partial r}, \qquad \frac{df}{dz} = \frac{1}{iz}\frac{\partial f}{\partial \theta}.$$

(ii) Now, calling $f = u + iv$, deduce that

$$\partial_r u = \frac{1}{r}\partial_\theta v\,, \qquad \frac{1}{r}\partial_\theta u = -\partial_r v\,.$$

(iii) These relations are just Cauchy–Riemann in polar coordinates. Check that they yield Laplace's equation $\Delta u = 0 = \Delta v$ where $\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$.
(iv) Verify that the function $u(r,\theta) = r^2\cos(2\theta)$ is harmonic. Use the polar form of Cauchy–Riemann to find the conjugate harmonic function $v(r,\theta)$. Express your final answer $f = u + iv$ in terms of z.