Homework 8

due March 12, 2004

Question 1. Use uniform convergence of $1/(1-z) = \sum_{n=0}^{\infty} z^n$ on $|z| \le R < 1$ to derive power series expansions for $\log(1-z)$ and $1/(1-z)^2$.

Question 2. Find Laurent series for the following functions in the regions indicated

(i) $f(z) = \dfrac{z}{(z-1)(z-3)}$ for $0 < |z-1| < 2$

(ii) $f(z) = \dfrac{16}{z^2(z-4)}$ for $0 < |z| < 4$ and $|z| > 4$

Question 3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converge for $|z| < R$. If $0 < r < R$, show that $f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$, where $z = re^{i\theta}$ and

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

Also show

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

The second equation is known as **Parseval's theorem**.
Hint: Expand $f\overline{f}$ in a series and integrate term by term.

Question 4. Evaluate

$$\oint_\gamma \frac{z^2 + e^z}{z(z-3)} dz$$

where γ is the unit circle.

Question 5. Find and classify the singularities of each of the following functions:

(i) $\dfrac{z^3+1}{z^2(z+1)}$

(ii) $z^3 e^{1/z}$

(iii) $\dfrac{\cos z}{z^2+1}$

(iv) $\dfrac{1}{e^z - 1}$