Homework Set 3: Exercises on Linear Spans and Bases

Directions: Please work on all exercises! Hand in Problems 1 and 2 as your "Calculational Homework" and Problems 5 and 7 as your "Proof-Writing Homework" at the beginning of lecture on January 26, 2007.

As usual, we are using $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

1. Show that the vectors $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$, and $v_3 = (2, -1, 1)$ are linearly independent in $\mathbb{R}^3$. Write the vector $v = (1, -2, 5)$ as a linear combination of v_1, v_2, and v_3.

2. Consider the complex vector space $V = \mathbb{C}^3$ and the list (v_1, v_2, v_3) of vectors in V, where $v_1 = (i, 0, 0)$, $v_2 = (i, 1, 0)$, and $v_3 = (i, i, -1)$. Show that $\text{span}(v_1, v_2, v_3) = V$.

3. Find a basis for the subspace U of $\mathbb{R}^5$ defined by
$$U = \{(x_1, x_2, \ldots, x_5) \mid x_1 = 3x_2, x_3 = 7x_4\}.$$

4. Let V be a vector space over $\mathbb{F}$, and suppose that the list $(v_1, v_2, \ldots, v_n)$ of vectors spans V, where each $v_i \in V$. Prove that the list
$$(v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots, v_{n-2} - v_{n-1}, v_{n-1} - v_n, v_n)$$
also spans V.

5. Let V be a vector space over $\mathbb{F}$, and suppose that $(v_1, v_2, \ldots, v_n)$ is a linearly independent list of vectors in V. Given any $w \in V$ such that
$$(v_1 + w, v_2 + w, \ldots, v_n + w)$$
is a linearly dependent list of vectors in V, prove that $w \in \text{span}(v_1, v_2, \ldots, v_n)$.

6. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $\dim(V) = n$ for some $n \in \mathbb{Z}_+$. Prove that there are n one-dimensional subspaces $U_1, U_2, \ldots, U_n$ of V such that
$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n.$$

7. Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose that U is a subspace of V for which $\dim(U) = \dim(V)$. Prove that $U = V$.

8. Let $\mathbb{F}_m[z]$ denote the vector space of all polynomials with degree less than or equal to $m \in \mathbb{Z}_+$ and having coefficient over $\mathbb{F}$, and suppose that $p_0, p_1, \ldots, p_m \in \mathbb{F}_m[z]$ satisfy $p_j(2) = 0$. Prove that $(p_0, p_1, \ldots, p_m)$ is a linearly dependent list of vectors in $\mathbb{F}_m[z]$.

9. Let U and V be five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

10. Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose that $U_1, U_2, \ldots, U_m$ are any m subspaces of V. Prove that
$$\dim(U_1 + U_2 + \cdots + U_m) \leq \dim(U_1) + \dim(U_2) + \cdots + \dim(U_m).$$