Homework Set 5: Exercises on Matrices and Linear Maps

Directions: Please work on all problems! Hand in solutions to the Calculational Problems 1, 2(i,m,r), 5(a), 6(a) and the Proof-Writing Problems 11 and 13 at the **beginning** of lecture on February 9, 2007.

As usual, we are using $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

1. Suppose that A, B, C, D, and E are matrices over $\mathbb{F}$ having the following sizes:

$$A \text{ is } 4\times 5, \quad B \text{ is } 4\times 5, \quad C \text{ is } 5\times 2, \quad D \text{ is } 4\times 2, \quad E \text{ is } 5\times 4.$$

   Determine whether the following matrix expressions are defined, and, for those that are defined, determine the size of the resulting matrix.

   (a) BA (b) $AC+D$ (c) $AE+B$ (d) $AB+B$ (e) $E(A+B)$ (f) $E(AC)$

2. Suppose that A, B, C, D, and E are the following matrices:

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

   Determine whether the following matrix expressions are defined, and, for those that are defined, compute the resulting matrix.

   (a) $D+E$ (b) $D-E$ (c) $5A$ (d) $-7C$ (e) $2B-C$ (f) $2E-2D$
   (g) $-3(D+2E)$ (h) $A-A$ (i) AB (j) BA (k) $(3E)D$ (l) $(AB)C$
   (m) $A(BC)$ (n) $(4B)C+2B$ (o) $D-3E$ (p) $CA+2E$ (q) $4E-D$ (r) DD

3. Suppose that A, B, and C are the following matrices and that $a = 4$ and $b = -7$.

$$A = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}.$$

   Verify computationally that

   (a) $A+(B+C) = (A+B)+C$ (b) $(AB)C = A(BC)$ (c) $(a+b)C = aC+bC$
   (d) $a(B-C) = aB-aC$ (e) $a(BC) = (aB)C = B(aC)$ (f) $A(B-C) = AB-AC$
   (g) $(B+C)A = BA+CA$ (h) $a(bC) = (ab)C$ (i) $B-C = -C+B$

4. Suppose that A is the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}.$$

Compute $p(A)$ where $p(x)$ is given by

(a) $p(x) = x - 2$ (b) $p(x) = 2x^2 - x + 1$ (c) $p(x) = x^3 - 2x + 4$ (d) $p(x) = x^2 - 4x + 1$

5. In each of the following, find matrices A, x, and b such that the given system of linear equations can be expressed as the single matrix equation $Ax = b$.

$$\text{(a)} \left\{ \begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 5x_3 & = & 7 \\ 9x_1 & - & x_2 & + & x_3 & = & -1 \\ x_1 & + & 5x_2 & + & 4x_3 & = & 0 \end{array} \right. \qquad \text{(b)} \left\{ \begin{array}{rcrcrcrcr} 4x_1 & & & - & 3x_3 & + & x_4 & = & 1 \\ 5x_1 & + & x_2 & & & - & 8x_4 & = & 3 \\ 2x_1 & - & 5x_2 & + & 9x_3 & - & x_4 & = & 0 \\ & & 3x_2 & - & x_3 & + & 7x_4 & = & 2 \end{array} \right.$$

6. In each of the following, express the matrix equation as a system of linear equations.

$$\text{(a)} \begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 3 & -2 & 0 & 1 \\ 5 & 0 & 2 & -2 \\ 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

7. Let U, V, and W be finite-dimensional vector spaces over $\mathbb{F}$ with $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Prove that

$$\dim(\text{null}(T \circ S)) \leq \dim(\text{null}(T)) + \dim(\text{null}(S)).$$

8. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $S, T \in \mathcal{L}(V, V)$. Prove that $T \circ S$ is invertible if and only if both S and T are invertible.

9. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $S, T \in \mathcal{L}(V, V)$, and denote by I the identity map on V. Prove that $T \circ S = I$ if and only if $S \circ T = I$.

10. Let $n \in \mathbb{Z}_+$ be a positive integer and $a_{i,j} \in \mathbb{F}$ be scalars for $i, j = 1, \ldots, n$. Prove that the following two statements are equivalent:

(a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\sum_{k=1}^{n} a_{1,k}x_k = 0$$

$$\vdots$$

$$\sum_{k=1}^{n} a_{n,k}x_k = 0.$$

(b) For every choice of scalars $c_1, \ldots, c_n \in \mathbb{F}$, there is a solution to the system of equations

$$\sum_{k=1}^{n} a_{1,k}x_k = c_1$$

$$\vdots$$

$$\sum_{k=1}^{n} a_{n,k}x_k = c_n.$$

11. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $T \in \mathcal{L}(V, V)$, and let $U_1, \ldots, U_m$ be subspaces of V that are invariant under T. Prove that $U_1 + \cdots + U_m$ must then also be an invariant subspace of V under T.

12. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $T \in \mathcal{L}(V, V)$, and suppose that U_1 and U_2 are subspaces of V that are invariant under T. Prove that $U_1 \cap U_2$ is also an invariant subspace of V under T.

13. Let $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^2)$ be defined by

$$T(u, v) = (v, u)$$

for every $u, v \in \mathbb{F}$. Compute the eigenvalues and associated eigenvectors for T.

14. Let $T \in \mathcal{L}(\mathbb{F}^3, \mathbb{F}^3)$ be defined by

$$T(u, v, w) = (2v, 0, 5w)$$

for every $u, v, w \in \mathbb{F}$. Compute the eigenvalues and associated eigenvectors for T.

15. Let $n \in \mathbb{Z}_+$ be a positive integer and $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ be defined by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$$

for every $x_1, \ldots, x_n \in \mathbb{F}$. Compute the eigenvalues and associated eigenvectors for T.

16. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $T \in \mathcal{L}(V, V)$ invertible and $\lambda \in \mathbb{F} \setminus \{0\}$. Prove that λ is an eigenvalue for T if and only if λ^{-1} is an eigenvalue for T^{-1}.

17. Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, V)$ has the property that every $v \in V$ is an eigenvector for T. Prove that T must then be a scalar multiple of the identity function on V.