# Homework Set 6: Exercises on Eigenvalues

**Directions**: Please work on all exercises and hand in your solutions to Problems 6 and 7 at the **beginning** of lecture on February 16, 2006. (Because of the midterm there is only one set of homeworks this week!)

As usual, we are using $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{F}[z]$ denotes the set of polynomials with coefficients over $\mathbb{F}$.

1. Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, and let $S, T \in \mathcal{L}(V)$ be linear operators on $V$ with $S$ invertible. Given any polynomial $p(z) \in \mathbb{F}[z]$, prove that
$$p(S \circ T \circ S^{-1}) = S \circ p(T) \circ S^{-1}.$$

2. Let $V$ be a finite-dimensional vector space over $\mathbb{C}$, $T \in \mathcal{L}(V)$ be a linear operator on $V$, and $p(z) \in \mathbb{C}[z]$ be a polynomial. Prove that $\lambda \in \mathbb{C}$ is an eigenvalue of the linear operator $p(T) \in \mathcal{L}(V)$ if and only if $T$ has an eigenvalue $\mu \in \mathbb{C}$ such that $p(\mu) = \lambda$.

3. Let $V$ be a finite-dimensional vector space over $\mathbb{C}$ with $T \in \mathcal{L}(V)$ a linear operator on $V$. Prove that, for each $k = 1, \ldots, \dim(V)$, there is an invariant subspace $U_k$ of $V$ under $T$ such that $\dim(U_k) = k$.

4. Prove or give a counterexample to the following claim:

   **Claim.** *Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, and let $T \in \mathcal{L}(V)$ be a linear operator on $V$. If the matrix for $T$ with respect to some basis on $V$ has all zeros on the diagonal, then $T$ is not invertible.*

5. Prove or give a counterexample to the following claim:

   **Claim.** *Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, and let $T \in \mathcal{L}(V)$ be a linear operator on $V$. If the matrix for $T$ with respect to some basis on $V$ has all non-zero elements on the diagonal, then $T$ is invertible.*

6. Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, and let $S, T \in \mathcal{L}(V)$ be linear operators on $V$. Suppose that $T$ has $\dim(V)$ distinct eigenvalues and that, given any eigenvector $v$ for $T$ associated to some eigenvalue $\lambda$, $v$ is also an eigenvector for $S$ associated to some (possibly distinct) eigenvalue $\mu$. Prove that $T \circ S = S \circ T$.

7. Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, and suppose that the linear operator $P \in \mathcal{L}(V)$ has the property that $P^2 = P$. Prove that $V = \operatorname{null}(P) \oplus \operatorname{range}(P)$.