Introduction to Complex Numbers. Summary

Isaiah Lankham, Bruno Nachtergaele, Anne Schilling
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1 Definition of Complex Numbers

Let $\mathbb{R}$ denote the set of *real numbers*. We will denote the set of *complex numbers* by $\mathbb{C}$. Here is the definition.

Definition 1.1. The set of *complex numbers* $\mathbb{C}$ is defined as

$$\mathbb{C} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

For any complex number $z = (x, y)$, we call $\mathrm{Re}(z) = x$ the *real part* of z and $\mathrm{Im}(z) = y$ the *imaginary part* of z.

In other words, we are defining a new collection of numbers z by taking every possible ordered pair (x, y) of real numbers $x, y \in \mathbb{R}$, and x is called the real part of the ordered pair (x, y) to imply that the set of real numbers $\mathbb{R}$ should be identified with the subset $\{(x, 0) \mid x \in \mathbb{R}\} \subset \mathbb{C}$. It is also common to use the term *purely imaginary* for any complex number of the form $(0, y)$, where $y \in \mathbb{R}$. In particular, the complex number $(0, 1)$ is special, and it is given the name *imaginary unit*. It is standard to denote it by the single letter i (or j if i is being used for something else, such as for electric current in Electrical Engineering).

Note that $z = (x, y) = x(1, 0) + y(0, 1) = x1 + yi$. We usually write $z = x + iy$. It is often significantly easier to perform arithmetic operations on complex numbers when written in "$x + iy$" notation, rather than the ordered pair notation of the definition.

2 Operations on Complex Numbers

2.1 Addition and Subtraction of Complex Numbers

Addition of complex numbers is performed component-wise, meaning that the real and imaginary parts are simply combined.

Definition 2.1. Given two complex numbers $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$, we define their *(complex) sum* to be

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Example 2.2. $(3, 2) + (17, -4.5) = (3 + 17, 2 - 4.5) = (20, -2.5)$.

As with the real numbers, subtraction is defined as addition of the opposite number, a.k.a. the *additive inverse* of $z = (x, y)$, which is defined as $-z = (-x, -y)$.

Example 2.3. $(\pi, \sqrt{2}) - (\pi/2, \sqrt{19}) = (\pi - \pi/2, \sqrt{2} - \sqrt{19}) = (\pi/2, \sqrt{2} - \sqrt{19})$.

The addition of complex numbers shares a few other properties with the addition of real numbers, including associativity, commutativity, the existence and uniqueness of the additive identity (or neutral element) denoted by "0", and the existence and uniqueness of the additive inverse already mentioned above. We summarize these properties in Theorem 2.4 below.

Theorem 2.4. *Let $z_1, z_2, z_3 \in \mathbb{C}$ be any three complex numbers. Then the following statements are true.*

1. *(Associativity)* $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
2. *(Commutativity)* $z_1 + z_2 = z_2 + z_1$.
3. *(Additive Identity) There is a unique complex number denoted* 0 *such that* $0 + z_1 = z_1$. *Moreover,* $0 = (0, 0)$.
4. *(Additive Inverses) Given $z \in \mathbb{C}$, there is a unique complex number denoted $-z$ such that $z + (-z) = 0$. Moreover, if $z = (x, y)$ with $x, y \in \mathbb{R}$, then $-z = (-x, -y)$.*

The proof of this Theorem is straightforward. Just use the definition of $+$ (when used to denote the addition of complex numbers) and the familiar properties of the addition of real numbers. The properties in Theorem 2.4 are collectively called the properties of a *commutative group*. Another word for commutative is *abelian*. So, we say that $\mathbb{C}$ is a commutative group (a.k.. an abelian group) under the operation of addition. Note that $+$ can be regarded as a function from $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$. Such a function is often called a *binary operation*.

2.2 Multiplication of Complex Numbers

The definition of multiplication for two complex numbers is at first glance somewhat less straightforward than that of addition. However, it naturally follows

Definition 2.5. Given two complex numbers $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$, we define their *(complex) product* to be

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

According to this definition $i^2 = -1$. In other words, i is a solution of the polynomial equation $z^2 + 1 = 0$, which does not have solutions in $\mathbb{R}$. This was originally the main motivation for introducing the complex numbers. Note that the relation $i^2 = -1$ and assumption that real complex numbers multiply as real numbers do, and that the other basic properties of real number arithmetic apply to complex numbers, is sufficient to arrive at the general rule for multiplication of complex numbers

$$\begin{aligned}(x_1 + y_1 i)(x_2 + y_2 i) &= x_1x_2 + x_1y_2 i + x_2y_1 i + y_1y_2 i^2\\ &= x_1x_2 + x_1y_2 i + x_2y_1 i - y_1y_2\\ &= x_1x_2 - y_1y_2 + (x_1y_2 + x_2y_1)i\end{aligned}$$

As with addition, the basic properties of complex multiplication are easy to prove enough using the definition. We summarize these properties in Theorem 2.6 below.

Theorem 2.6. *Let $z_1, z_2, z_3 \in \mathbb{C}$ be any three complex numbers. Then the following statements are true.*

1. *(Associativity)* $(z_1z_2)z_3 = z_1(z_2z_3)$.
2. *(Commutativity)* $z_1z_2 = z_2z_1$.
3. *(Multiplicative Identity) There is a unique complex number denoted* 1 *such that* $1z_1 = z_1$. *Moreover,* $1 = (1, 0)$.
4. *(Distributivity of Multiplication over Addition)* $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

Just as is the case for real numbers, any non-zero complex number z has a unique multiplicative inverse, which we may denote by z^{-1} or $1/z$.

Theorem 2.6 (continued).

5. *(Multiplicative Inverses) Given* $z \in \mathbb{C}$ *such that* $z \neq (0, 0)$, *there is a unique complex number denoted* z^{-1} *such that* $zz^{-1} = 1$. *Moreover, if* $z = (x, y)$ *with* $x, y \in \mathbb{R}$, *then*

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right).$$

Proof. Uniqueness. A complex number w is an inverse of z if $zw = 1$ (by the commutativity of complex multiplication this is equivalent to $wz = 1$). We will first prove that if w and v are two complex numbers, such that $zw = 1$ and $zv = 1$, then we necessarily have $w = v$. This means that any $z \in \mathbb{C}$ can have at most one inverse. To see this, we start from $zv = 1$. By multiplying both sides by w, we obtain $wzv = w1$. Using the fact that 1 is the multiplicative unit, the commutativity of the product, and the assumption that w is an inverse, we get $zwv = v = w$.

Existence. Now assume $z \in \mathbb{C}$ with $z \neq 0$, and write $z = x + iy$, with $x, y \in \mathbb{R}$. Since $z \neq 0$, at least one of x or y does not vanish. Hence $x^2 + y^2 > 0$. Therefore, we can define

$$w = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right),$$

and you can check that $zw = 1$ by a straightforward computation. □

Now, we can define the division of a complex number z_1 by a non-zero complex number z_2 as the product of z_1 and z_2^{-1}. Explicitly, for two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have that their *(complex) quotient* is

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2 + (x_2y_1 - x_1y_2)\, i}{x_2^2 + y_2^2}.$$

Example 2.7. We illustrate the above definition with the following example:

$$\frac{(1,2)}{(3,4)} = \left(\frac{1 \cdot 3 + 2 \cdot 4}{3^2 + 4^2}, \frac{3 \cdot 2 - 1 \cdot 4}{3^2 + 4^2} \right) = \left(\frac{3+8}{9+16}, \frac{6-4}{9+16} \right) = \left(\frac{11}{25}, \frac{2}{25} \right).$$

2.3 Complex Conjugate

Complex conjugation is an operation on complex numbers without analogue in the real numbers (it acts trivially on real numbers). Nonetheless, it will turn out to be very useful.

Definition 2.8. Given a complex number $z = (x, y) \in \mathbb{C}$ with $x, y \in \mathbb{R}$, we define the *(complex) conjugate* of z to be the complex number

$$\bar{z} = (x, -y).$$

The following properties for the complex conjugation are easy to prove.

Theorem 2.9. *Given two complex numbers $z_1, z_2 \in \mathbb{C}$,*

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
2. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.
3. $\overline{1/z_1} = 1/\overline{z_1}$, *for all* $z_1 \neq 0$.
4. $\overline{z_1} = z_1$ *if and only if* $\mathrm{Im}(z_1) = 0$.

5. $\overline{\overline{z_1}} = z_1$.

6. *the real and imaginary parts of* z_1 *can be expressed as*

$$\operatorname{Re}(z_1) = \frac{1}{2}(z_1 + \overline{z_1}) \quad \text{and} \quad \operatorname{Im}(z_1) = \frac{1}{2i}(z_1 - \overline{z_1}).$$

2.4 The Modulus (a.k.a. Norm, Length, or Magnitude)

In this section, we introduce yet another operation on complex numbers, this time based upon a generalization of the notion of *absolute value* of a real number. To motivate the definition, it is useful to view the set of complex numbers as the two-dimensional Euclidean plane, i.e., to think of $\mathbb{C} = \mathbb{R}^2$ being equal as sets. The *modulus*, or *length*, of $z \in \mathbb{C}$ is then defined as the Euclidean distance between z, as a point in the plane, and the origin $(0, 0)$. This is the content of the following definition.

Definition 2.10. Given a complex number $z = (x, y) \in \mathbb{C}$ with $x, y \in \mathbb{R}$, the *modulus* of z is defined to be

$$|z| = \sqrt{x^2 + y^2}$$

In particular, note that, given $x \in \mathbb{R}$,

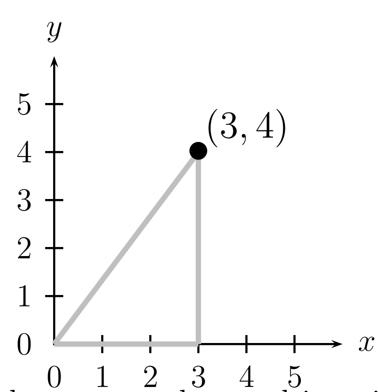
$$|(x, 0)| = \sqrt{x^2 + 0} = |x|$$

under the convention that the square root function takes on its principal positive value.

Example 2.11. To see geometrically that the modulus of the complex number $(3, 4)$ is

$$|(3, 4)| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5,$$

construct the following diagram in the Euclidean plane



and apply the Pythagorean theorem to the resulting right triangle in order to find the distance from the origin to the point $(3, 4)$.

The following theorem lists some fundamental properties of the modulus.

Theorem 2.12. *Given two complex numbers $z_1, z_2 \in \mathbb{C}$,*

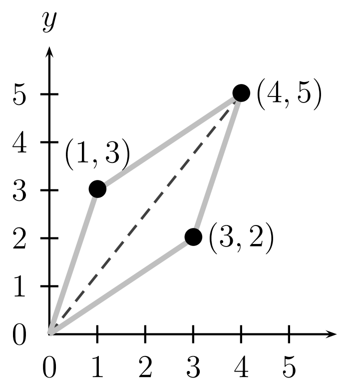
1. $|z_1 z_2| = |z_1| \cdot |z_2|$.
2. $|z_1/z_2| = |z_1|/|z_2|$ *assuming that* $z_2 \neq 0$.
3. $|\overline{z_1}| = |z_1|$.
4. $|\mathrm{Re}(z_1)| \leq |z_1|$ *and* $|\mathrm{Im}(z_1)| \leq |z_1|$.
5. *(Triangle Inequality)* $|z_1 + z_2| \leq |z_1| + |z_2|$.
6. *(Another Triangle Inequality)* $|z_1 - z_2| \geq |\,|z_1| - |z_2|\,|$.
7. *(Formula for Multiplicative Inverse)* $z_1\overline{z_1} = |z_1|^2$, *from which* $z_1^{-1} = \overline{z_1}/|z_1|^2$ *assuming* $z_1 \neq 0$.

2.5 Complex Numbers as Vectors in $\mathbb{R}^2$

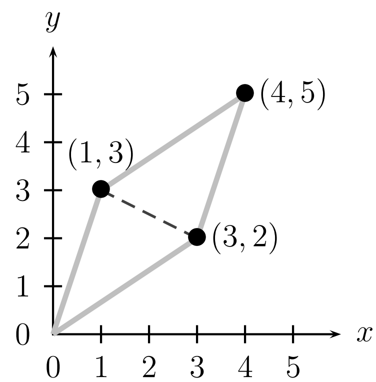
When complex numbers are viewed as points in the Euclidean plane $\mathbb{R}^2$, several of the operations defined in Section 2 can be directly visualized as if they were operations on *vectors*. For the purposes of these notes, we think of vectors as directed line segments that start at the origin and end at a specified point in the Euclidean plane. These line segments may also be moved around in space as long as the direction (which we will call the *argument* in Section 3.1 below) and the length (a.k.a. the modulus) are preserved. As such, the distinction between points in the plane and vectors is merely a matter of convention as long as we at least implicitly think of each vector as having been translated so that it starts at the origin.

As we saw in Section 2.4 above, the complex modulus can be viewed as the length of the hypotenuse of a certain right triangle. The sum and difference of two vectors can also each be represented geometrically as the lengths of specific diagonals within a particular parallelogram that is formed by copying and translating the two vectors being combined.

Example 2.13. We illustrate the sum $(3, 2) + (1, 3) = (4, 5)$ as the main, dashed diagonal of the parallelogram in the left-most figure below. The difference $(3, 2) - (1, 3) = (2, -1)$ can also be viewed as the shorter diagonal of the same parallelogram after is has been translated in order to begin at the origin. The latter is illustrated in the right-most figure below.



Vector Sum as Main Diagonal of Parallelogram



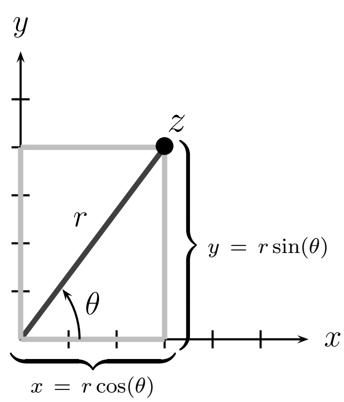
Vector Difference as Minor Diagonal of Parallelogram

3 Polar Form and Geometric Interpretation for $\mathbb{C}$

As mentioned above, $\mathbb{C}$ coincides with the plane $\mathbb{R}^2$ when viewed as a set of ordered pairs of real numbers. Therefore, we can use *polar coordinates* as an alternate way to uniquely identify a complex number. This gives rise to the so-called *polar form* for a complex number, which turns out to be an often very convenient representation of complex numbers.

3.1 Polar Form for Complex Numbers

The following diagram summarizes the relations between cartesian and polar coordinates in $\mathbb{R}^2$.



We call the ordered pair (x, y) the *rectangular coordinates* for the complex number z.

We also call the ordered pair (r, θ) the *polar coordinates* for the complex number z. The radius $r = |z|$ is called the *modulus* of z (as defined in Section 2.4 above), and the angle $\theta = \text{Arg}(z)$ is called the *argument* of z. Since the argument of a complex number describes an angle that is measured relative to the x-axis, it is important to note that θ is only well-defined up to adding multiples of 2π. As such, we restrict $\theta \in [0, 2\pi)$ and add or subtract multiples of 2π as needed (e.g., when multiplying two complex numbers so that their arguments are added together) in order to keep the argument within this range of values.

It is straightforward to transform polar coordinates into rectangular coordinates using the equations

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

In order to transform rectangular coordinates into polar coordinates, we first note that $r = \sqrt{x^2 + y^2}$ is just the complex modulus. Then θ must be chosen so that it satisfies the bounds $0 \leq \theta < 2\pi$ in addition to the simultaneous equations

$$x = \cos(\theta) \quad \text{and} \quad y = \sin(\theta),$$

where we are assuming that $z \neq 0$.

Summarizing:

$$z = x + yi = r\cos(\theta) + r\sin(\theta)i = r(\cos(\theta) + \sin(\theta)i).$$

Part of the utility of this expression is that the size $r = |z|$ of z is explicitly part of the very definition since it is easy to check that $|\cos(\theta) + \sin(\theta)i| = 1$ for any choice of $\theta \in \mathbb{R}$.

Closely related is the *exponential form* for complex numbers, which does nothing more than replace the expression $\cos(\theta) + \sin(\theta)i$ with $e^{i\theta}$. The real power of this definition is that this exponential notation turns out to be completely consistent with the usual usage of exponential notation for real numbers.

3.2 Geometric Multiplication for Complex Numbers

As alluded to in Section 3.1 above, the general exponential form for a complex number z is an expression of the form $re^{i\theta}$ where r is a non-negative real number and $\theta \in [0, 2\pi)$. The utility of this notation is immediately observed upon multiplying two complex numbers and applying rules for working with exponents that can be proven to remain true in this significantly more abstract setting:

Lemma 3.1. *Let $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2} \in \mathbb{C}$. Then*

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}.$$

Proof. We have

$$\begin{aligned} z_1z_2 &= (r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i\theta_1}e^{i\theta_2} \\ &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2\big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big] \\ &= r_1r_2\big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big] = r_1r_2e^{i(\theta_1+\theta_2)} \end{aligned}$$

where we have used the usual formulas for sin and cos for the sum of angles. $\square$

In particular Lemma 3.1 shows that the modulus $|z_1 z_2|$ of the product is the product of the moduli r_1 and r_2 and that the argument $\mathrm{Arg}(z_1 z_2)$ of the product is the sum of the arguments $\theta_1 + \theta_2$.

3.3 Exponentiation and Root Extraction

Another important use for the polar form of a complex number is in exponentiation. The simplest possible situation here involves the use of a positive integer as a power, in which case exponentiation is nothing more than repeated multiplication. Given the observations in Section 3.2 above and using some trigonometric identities, one quickly obtains the following well-known result.

Theorem 3.2 (de Moivre's Formula)**.** *Let $z = r(\cos(\theta) + \sin(\theta)i)$ be a complex number in polar form and $n \in \mathbb{Z}_+$ be a positive integer. Then*

1. *the exponentiation $z^n = r^n(\cos(n\theta) + \sin(n\theta)i)$ and*
2. *the n^{th} roots of z are given by the n complex numbers*

$$z_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + \sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) i\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

Note in particular that we are not only always guaranteed the existence of an n^{th} root for any complex number, but that we are also always guaranteed to have exactly n of them. This level of completeness in root extraction contrasts very sharply with the delicate care that must be taken when one wishes to extract roots of real numbers without the aid of complex numbers.

An important special case of de Moivre's Formula yields an infinite family of well-studied of numbers called the *roots of unity*. By *unity* we just mean the complex number $1 = 1 + 0i$, and by the n^{th} *roots of unity* we mean the n numbers

$$\begin{aligned} z_k &= 1^{1/n}\left[\cos\left(\frac{0}{n} + \frac{2\pi k}{n}\right) + \sin\left(\frac{0}{n} + \frac{2\pi k}{n}\right) i\right] \\ &= \cos\left(\frac{2\pi k}{n}\right) + \sin\left(\frac{2\pi k}{n}\right) i \\ &= e^{2\pi i(k/n)}, \end{aligned}$$

where $k = 0, 1, 2, \ldots, n-1$. These numbers have many interesting properties and important applications despite how simple they might appear to be.

3.4 Some Complex Elementary Functions

We conclude these notes by defining three of the basic elementary functions that take complex arguments. In this context, "elementary function" is used as a technical term and essentially means something like "one of the most common forms of function encountered when beginning to learn Calculus." The most basic elementary functions include the familiar polynomial and algebraic functions, such as the n^{th} root function, in addition to the somewhat more sophisticated exponential function, the trigonometric functions, and the logarithmic function. For the purposes of these notes, we will now define the complex exponential function and two complex trigonometric functions. However, definitions for the remaining basic elementary functions can be found in any book on Complex Analysis.

The basic groundwork for defining the *complex exponential function* was already put into place in Sections 3.1 and 3.2 above. In particular, we have already defined the expression $e^{i\theta}$ to mean the sum $\cos(\theta) + \sin(\theta)i$ for any real number θ. Historically, this equivalence is a special case of the more general *Euler's formula*

$$e^{x+yi} = e^x(\cos(y) + \sin(y)i),$$

which we here take as our definition of the complex exponential function applied to any complex number $x + yi$ for $x, y \in \mathbb{R}$.

Given this exponential function, one can them define the *complex sine function* and the *complex cosine function* as

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

Remarkably, these functions retain many of their familiar properties, which should be taken as a sign that the definitions — however abstract — have been well thought-out. We summarize a few of these properties as follows.

Theorem 3.3. *Given $z_1, z_2 \in \mathbb{C}$,*

1. $e^{z_1+z_2} = e^{z_1}e^{z_2}$ *and* $e^z \neq 0$ *for any choice of* $z \in \mathbb{C}$.
2. $\sin^2(z_1) + \cos^2(z_1) = 1$.
3. $\sin(z_1 + z_2) = \sin(z_1) \cdot \cos(z_2) + \cos(z_1) \cdot \sin(z_2)$.
4. $\cos(z_1 + z_2) = \cos(z_1) \cdot \cos(z_2) - \sin(z_1) \cdot \sin(z_2)$.