LECTURE 13: YANG–BAXTER EQUATION

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Recall from previous lecture the definition of the divided difference operator

$$\partial_i = \frac{1}{x_i - x_{i+1}}(1 - s_i).$$

We showed:

Proposition 0.1. *Let w_0 be the longest element, then*

$$\partial_{w_0} = a_\delta^{-1} \sum_{w \in S_n} \varepsilon(w) w,$$

where $a_\delta = \prod_{1 \le i < j \le n}(x_i - x_j)$ and $\delta = (n-1, n-2, \ldots, 1, 0)$

One can define

$$a_\alpha = \sum_{w \in S_n} \varepsilon(w) w(x^\alpha),$$

(α – any n-tuple of integers).
The Schur functions (Schur polynomials):

$$s_{\alpha-\delta} = \frac{a_\alpha}{a_\delta},$$

standard form:

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}.$$

Remark 0.2. $\partial_{w_0} x^\alpha = s_{\alpha-\delta}$ – the cause of using non-standard notation.

Definition 0.3. Define isobaric divided difference operators π_i:

$$\pi_i f = \partial_i(x_i f), \quad f \in \mathbb{Z}[x_1, \ldots, x_n].$$

This satisfies relations:

$$\pi_i \pi_j = \pi_j \pi_i, \qquad \text{if } |i-j| > 1,$$

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1},$$

$$\pi_i^2 = \pi_i.$$

Exercise: Check that these relations are satisfied.

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Remark 0.4. π_i are used to define Grothendieck polynomials similarly to ∂_i being used to define Schubert polynomials.

Asssociate to every permutation $w \in S_n$ an operator of degree 0:

$$\pi_w = \pi_{a_1} \dots \pi_{a_k}, \quad \text{where } a_1 \dots a_k \in \mathcal{R}(w)$$

Remark 0.5. This is independent of the reduced word since the graph $\Gamma(w)$ is connected and π_i satisfy the braid and commutation relations.

Proposition 0.6.

$$\pi_{w_0} f = a_\delta^{-1} \sum_{w \in S_n} \varepsilon(w) w(x^\delta f),$$

in particular,

$$\pi_{w_0} x^\alpha = s_\alpha.$$

Proof. $\pi_1 f = \partial_1(x_1 f)$, $\pi_1\pi_2 f = \partial_1(x_1\partial_2(x_2 f)) = (\text{can move } \partial_2 \text{ past } x_1) = \partial_1\partial_2(x_1 x_2 f)$,

....

$\pi_1 \dots \pi_r f = \partial_1 \dots \partial_r(x_1 \dots x_r f)$, also

$(1 \dots n-1)(1 \dots n-2) \dots (12)(1) \in \mathcal{R}(w_0)$

$\Rightarrow \pi_{w_0}(f) = \partial_{w_0}(x^\delta f)$

□

Definition 0.7. $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ – sets of indeterminates. Partial resultant

$$\Delta(x, y) = \prod_{i+j \le n} (x_i - y_j).$$

Definition 0.8. To each $w \in S_n$ associate a double Schubert polynomial $\sigma_w(x, y)$:

$$\sigma_w(x, y) = \partial_{w^{-1}w_0} \Delta(x, y),$$

where the divided difference operators are taken w.r.t. x variable. The simple Schubert polynomials are the specialization

$$\sigma_w(x) = \sigma_w(x, 0) = \partial_{w^{-1}w_0} x^\delta.$$

Remark 0.9. Thus there exist recursive formulas for Schubert polynomials:

$$\partial_u \sigma_w = \begin{cases} \sigma_{wu^{-1}}, & if\ l(wu^{-1}) = l(w) - l(u) \\ 0, & else \end{cases}$$

$$\partial_u \sigma_w = \partial_u \partial_{w^{-1}w_0} \Delta = \partial_{uw^{-1}w_0} \Delta,$$

if $l(uw^{-1}w_0) = l(u) + l(w^{-1}w_0)$ (see previous lecture, $= 0$ otherwise)

$$= \partial_{(wu^{-1})^{-1}w_0} \Delta = \sigma_{wu^{-1}}$$

$\Longleftrightarrow$

$$l(u) = l(uw^{-1}w_0) - l(w^{-1}w_0) = l(w_0) - l(uw^{-1}) - l(w_0) + l(w^{-1}).$$

1. Yang-Baxter Equations

(see Fomin, Kirillov, "The Yang-Baxter equations, symmetric functions and Schubert polynomials", Discrete Math. **153** *(1996), 123).*

Goal – combinatorial formulas for Schubert polynomials.

Definition 1.1. The Iwahori-Hecke algebra $\mathcal{H}_{ab}^n$ is generated by $u_1, \ldots, u_{n-1}$, satisfying the relations:

$$u_i u_j = u_j u_i, \ if \ |i-j| > 1,$$

$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1},$$

$$u_i^2 = a u_i + b.$$

Example 1.2. : $\mathcal{H}_{0,1}^n = \mathbb{C}[S_n]$,
$\mathcal{H}_{1,0}^n =$ algebra of isobaric divided differences,
$\mathcal{H} = \mathcal{H}_{0,0}^n$ – nil-Coxeter algebra. $\mathcal{H}$ has a basis indexed by permutations with multiplication rule:

$$u \cdot w = \begin{cases} uw, & \text{if } \ l(uw) = l(u) + l(w) \\ 0, & \text{else} \end{cases}$$

Set $h_i(x) = 1 + x u_i$.

Lemma 1.3.

$$h_i(x) h_i(y) = h(x+y),$$

$$h_i(x) h_j(y) = h_j(y) h_i(x), \ \ if \ |i-j| > 1,$$

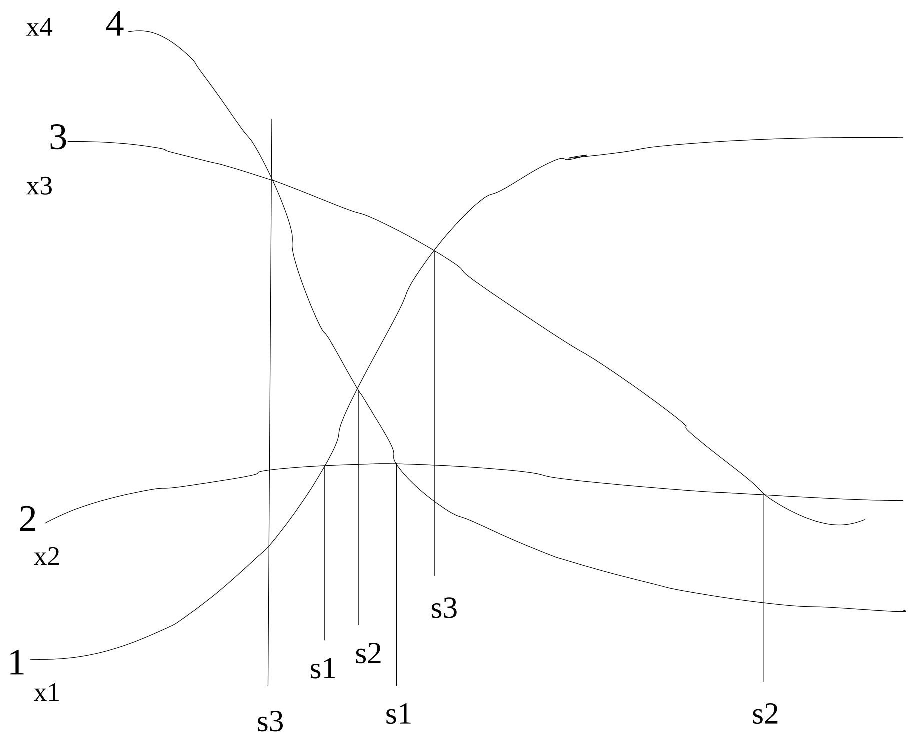
$$h_i(x) h_j(x+y) h_i(y) = h_j(y) h_i(x+y) h_j(x), \ \ |i-j| = 1$$

(the Yang-Baxter Equation).

Exercise: Check it.

Definition 1.4. A configuration is a family C of n continuous strands which cut each vertical line at a unique point.

Example 1.5. :



Each vertical line crosses every strand. Each pair of strands crosses at most once and at distinct x-coordinates.
$w = s_3 s_1 s_2 s_1 s_3 s_2$ is reduced since strands do not cross twice.

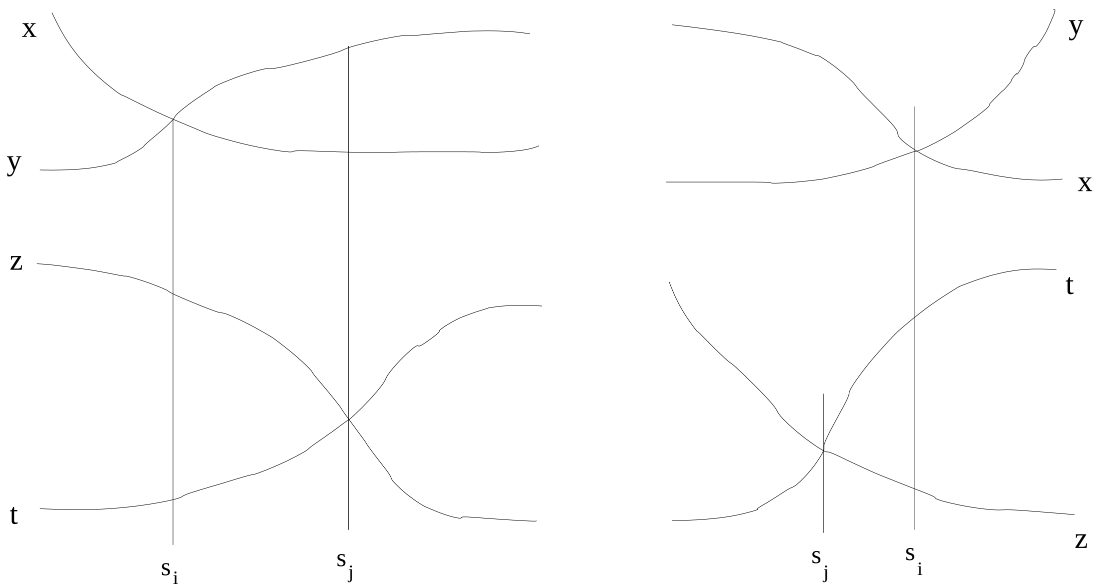
For $w = s_3 s_2 s_1 s_2 s_3 s_2$, $a_1 = 3$, $a_2 = 2$, $\ldots$, x_i – weight:

$$\phi(C) = h_{a_1}(x_{k_1} - x_{l_1}) h_{a_2}(x_{k_2} - x_{l_2}) \ldots h_{a_m}(x_{k_m} - x_{l_m})$$
(subtracted argument is weight of the strand with the steeper slope).

Lemma 1.6. *The weights of the strands being fixed, the polynomial $\phi(C)$ depends only on the permutation w underlying C.*

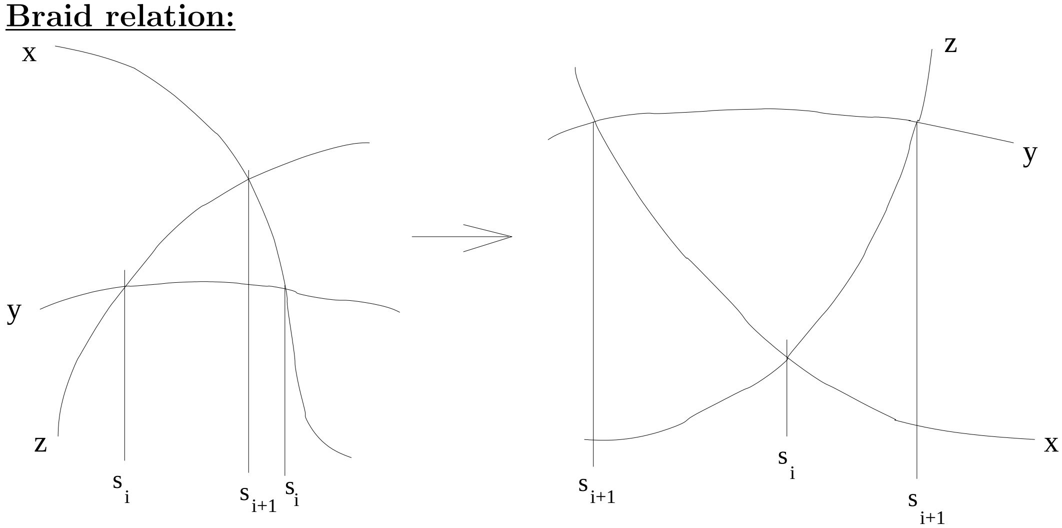
Proof. $\mathcal{G}(w)$ - graph of reduced words is connected. Hence it suffices to show that $\phi(C)$ remains unchanged under the commutation and braid relations.

Commutation relations:



$$h_i(x-y)h_j(z-t) = h_j(z-t)h_i(x-y),\ if\ |i-j|>1$$

Braid relation:



$$h_i(y-z)h_{i+1}(x-z)h_i(x-y) = h_{i+1}(x-y)h_i(x-z)h_{i+1}(y-z),$$

– the Yang-Baxter equation. □