LECTURE 16: COMBINATORIAL FORMULA FOR SINGLE SCHUBERT POLYNOMIALS AND RC-GRAPHS

JOSHUA CLEMENT

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1. Combinatorial formula for single Schubert polynomials

Theorem 1. *Combinatorial Theorem:*

$$\sigma_w\ (x) = \sum_{\underline{a}\ \in\ R(w)} \sum_{\underline{b}\ \in\ C(\underline{a})} x_{b_1}\ \dots\ x_{b_\ell}$$

where $C(\underline{a})$ *is the set of increasing* $\underline{a}$ *-compatible words,* ℓ *is the length of* w*, and*

(1) $b_1\ \leq\ b_2\ \leq\ \cdots\ \leq b_\ell$
(2) $b_i\ \leq\ a_i$
(3) $b_i < b_{i+1}$ *if* $a_i < a_{i+1}$

Proof. We have

$$\phi(\mathcal{C}_{\text{sp}})\ |_{y=0} = \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i) = \sigma(x)$$

(Recall: $h_i(x) = 1 + xu_i$, where the u_i's satisfy the nilCoxeter algebra) We need to expand the product and look for the coefficient of w; the b_i's are indices of the x's, and each h_{i+j-1} contributes u_{i+j-1}.

We get part (2) from the fact that $i\ \leq\ i + j - 1$, and we get (3) because since the product $\prod_{j=n-i}^{1}$ is decreasing, we must have $b_i < b_{i+1}$ if $a_i < a_{i+1}$. □

Example 2. Consider S_3. Then $\sigma(x) = h_2(x_1)h_1(x_1)h_2(x_2) = (1 + x_1u_2)(1 + x_1u_1)(1 + x_2u_2)$. Note that $(1 + x_1u_2)(1 + x_1u_1)$ from the term $i = 1$ in the inner product, which is decreasing, and $(1 + x_2u_2)$ comes from $i = 2$.

Aim 1: We want to prove that the Schubert polynomials $\sigma_w(x)$, $w\ \in\ S_\infty$, form an integral basis for $Z[x_1, x_2, \dots]$.
Aim 2: Monk's Rule—expansion of $\sigma_w\ \sigma_{s_i}$

2. rc-graphs

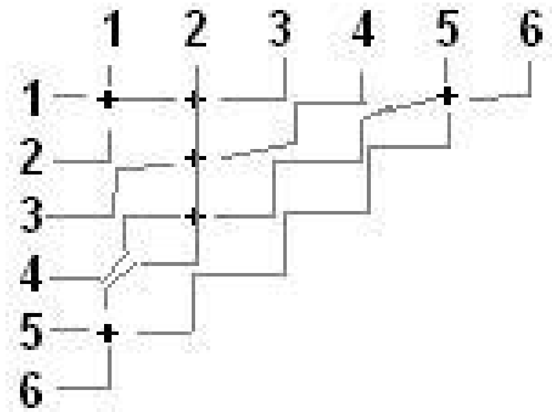
Reference: N. Bergeron, S. Billey, *RC graphs and Schubert Polynomials*, Exp. Math **2** (1993) 257-269

Definition 3. Let $\underline{a} = a_1a_2\ \dots\ a_p\ \in\ R(w)$ and $\underline{\alpha}\ = \alpha_1\ \dots\ \alpha_p\ \in\ C(\underline{a}\ )$. The reduced-word compatibel sequence graph or rc-graph for short is

$$D(\underline{a}, \underline{\alpha}) = \{(\alpha_k, a_k - \alpha_k + 1) \mid 1 \leq k \leq p\}.$$

Set

$$\mathcal{RC}(w) = \{D(\underline{a}, \underline{\alpha}) \mid \underline{a} \in R(w), \underline{\alpha} \in C(\underline{a})\}.$$



Example 4. $\underline{a} = 521345, \underline{\alpha} = 111235$

The plus signs indicate positions in $D(\underline{a}, \underline{\alpha})$; note that if $(i,j) \in D$, then $i + j \leq n$ if $w \in S_n$

Algorithm to get $w \in S_n$ from graph:

Each line alternates between going up and going to the right unless it hits a plus sign, in which case it goes through. Follow the strand labelled i from left to write to obtain $w(i)$.

In the example we have $w = [3, 1, 4, 6, 5, 2]$ (because $w(1) = 3$,$w(2) = 1$, etc.); $\ell(w) = 6$ since we have 6 crossings.

Note that strands do not cross more than once.

Remark 5. The transpose D^t of an rc-graph $D \in \mathcal{RC}(w)$ is an rc-graph in $\mathcal{RC}(w^{-1})$.

Denote by $\rho \colon \mathcal{RC}(w) \to \mathcal{RC}(w^{-1})$ the bijection mapping $D \mapsto D^t$.

Notation: For $D \in \mathcal{RC}(w)$ let $x_D = \prod_{(i,j) \in D} x_i$.

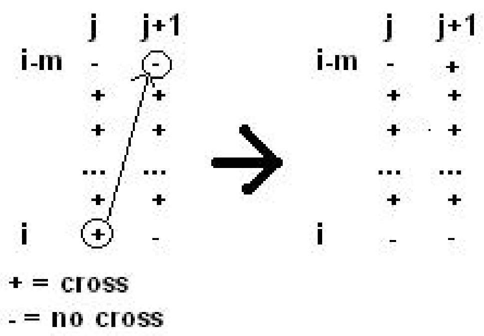
Corollary 6.

$$\sigma_w(x) = \Sigma_{D(\underline{a},\underline{\alpha}) \in \mathcal{RC}(w)} x_{D(\underline{a},\underline{\alpha})}$$

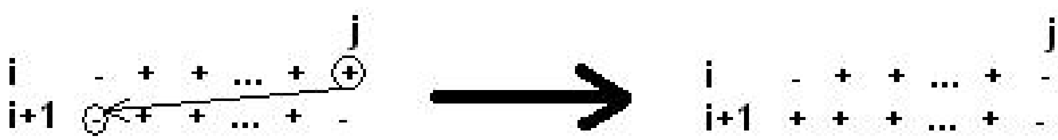
Date: February 11th, 2009.

3. Moves on rc-graphs

Let $w \in S_\infty$ and $D \in \mathcal{RC}(w)$. A ladder move L_{ij} is defined as:



A chute move C_{ij} is defined as:

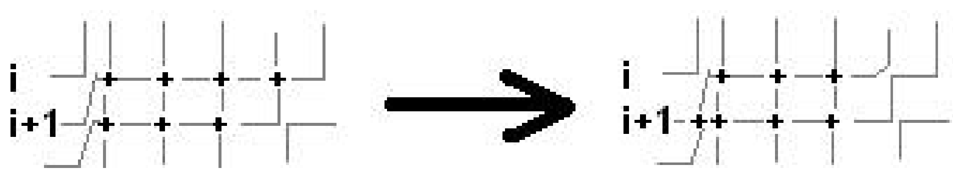


Remark 7. $\rho(L_{ij}(D)) = C_{ji}(\rho(D))$, i.e. L_{ij} and C_{ij} are dual to each other.

Lemma 8. *Ladder and chute moves preserve the permutation associated to D:*

$$\text{perm}C_{ij}(D) = \text{perm}(D)$$
$$\text{perm}L_{ij}(D) = \text{perm}(D)$$

Proof. We use a proof by picture. The strands in the region of a chute move look like this:

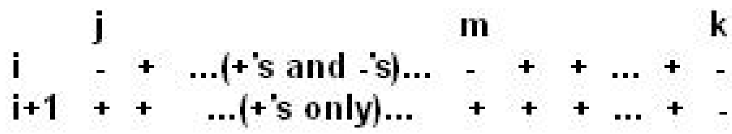


Following each strand one can easily check that each letter still gets mapped to the same position. □

Lemma 9. *$D \in \mathcal{RC}(w)$ is the result of a chute move (or, equivalently, admits an inverse chute move) if and only if there exists $(i, j) \notin D$ such that $(i + 1, j) \in D$.*

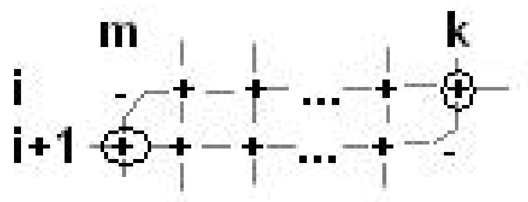
Remark 10. Geometrically, an inverse chute move cannot be applied if all +'s in each column are clumped at the top.

Proof. Suppose $(i,j) \notin D$ and $(i+1,j) \in D$. Look right along row $i+1$ for the smallest $k > j$ such that $(i+1,k) \notin D$ (There must be such a k since D contains only finitely many $+$).



Claim: $(i,k) \notin D$

Proof. Suppose this is not true, i.e. $(i,k) \in D$. Then we would have:



This is impossible because strands cannot cross twice. □

Let m be the position of the last dot before k, that is $m < k$ largest such that $(i,m) \notin D$. Therefore the $+$ at $(i+1,m)$ can be moved to (i,k) by an inverse chute move. □