LECTURE 20: THE AFFINE SYMMETRIC GROUP

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1. Recap from Last Lecture

Recall: $\tilde{S_n}$ is the affine symmetric group. Elements $\omega \in \tilde{S_n}$ are bijections from $\mathbb{Z}$ to itself satisfying:

(1) $\omega(i+n) = \omega(i) + n \quad \forall i \in \mathbb{Z}$
(2) $\sum_{i=1}^{n} \omega(i) = \binom{n+1}{2}$

Remark 1.1. For all $\omega \in \tilde{S_n}$ and $i, j \in \mathbb{Z}$, $\omega(i) \not\equiv \omega(j) \mod n \Longleftrightarrow i \not\equiv j \mod n$. This will be useful later.

2. Affine Inversion

Definition 2.1. The affine inversion number of $v \in \tilde{S_n}$ is

$$\widetilde{\mathrm{inv}}(v) = |\{(i,j) \in [n] \times \mathbb{Z} \mid i < j, v(i) > v(j)\}|.$$

Proposition 2.2. $\ell(v) = \widetilde{inv}(v)\ \forall v \in \tilde{S_n}$

Proof. Before we begin the proof of the proposition, we first give a claim:

Claim: $\widetilde{\mathrm{inv}}(v) \leq \ell(v)$.
Proof of claim.
It can be checked directly from the definition that for $1 \leq i \leq n-1$

$$(*) \quad \widetilde{\mathrm{inv}}(vs_i) = \begin{cases} \widetilde{\mathrm{inv}}(v) + 1 & \text{if } v(i) < v(i+1) \\ \widetilde{\mathrm{inv}}(v) - 1 & \text{if } v(i) > v(i+1). \end{cases}$$

The same is also true for $i = 0$ however, it is not obvious.

It is clear from the definition that (i,j) with $2 \leq i \leq n-1$ is an inversion of v if and only if $(i, s_0(j))$ is an inversion of vs_0. Hence it remains to consider the cases $i = 1$ and $i = n$.

Assume $v(n) < v(n+1)$. If $j > n+1$ and (n,j) is an inversion of v, then $(n, s_0(j))$ is an inversion of vs_0. Also, $(n, s_0(j))$ is an inversion of vs_0 and (n,j) is <u>not</u> an inversion of $v \Longleftrightarrow v(n+1) \geq v(j) \geq v(n)$. (†)

Similarly, if $j > 1$ and $(1,j)$ is an inversion of vs_0, then $(1, s_0(j))$ is an inversion of v. Also, $(1, s_0(j))$ is an inversion of vs_0 and $(1,j)$ is <u>not</u> and inversion of $vs_0 \Longleftrightarrow v(1) \geq v(s_0(j)) \geq v(0)$. (‡)

Since $v(i+n) = v(i) + n$, the cardinality of (†) and (‡) are equal.

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Note: $(n, n+1)$ is an inversion of vs_0 but not of v, which means that $\widetilde{\text{inv}}(vs_0) = \widetilde{\text{inv}}(v) + 1$.
Now, assume $v(n) > v(n+1)$.
By similar arguments as above, $\widetilde{\text{inv}}(vs_0) = \widetilde{\text{inv}}(v) - 1$. Therefore the recursion formula (*) for inversion has been proven.

To finish the proof of the claim notice that $\widetilde{\text{inv}}(e) = 0 = \ell(e)$. By the recursion formula we just proved, we have the $\widetilde{\text{inv}}(v) \leq \ell(v)$, thus proving the claim.

To prove the proposition, that $\widetilde{\text{inv}}(v) = \ell(v)$ we use induction on $\widetilde{\text{inv}}$.
If $\widetilde{\text{inv}}(v) = 0$, then we must have

$$v(1) < v(2) < \cdots < v(n) < v(n+1) = v(1) + n$$

which imples that $v = e$.

Next, suppose $\widetilde{\text{inv}}(v) = t + 1 > 0$ and assume by induction that $\widetilde{\text{inv}}(u) \leq t \Longrightarrow \widetilde{\text{inv}}(u) = \ell(u)$. Since $\widetilde{\text{inv}}(v) > 0$, we have $v \neq e \Longrightarrow \exists s \in S$ such that $\widetilde{\text{inv}}(vs) = t$.

Then, by the induction hypothesis, we have that

$$\widetilde{\text{inv}}(vs) = \ell(vs) = t \Longrightarrow \ell(v) \leq t + 1 \Longrightarrow \ell(v) \leq \widetilde{\text{inv}}(v).$$

Therefore we have shown that $\widetilde{\text{inv}}(v) = \ell(v)$. □

A consequence of the previous result is a simple description of the descent set of affine permutations.

Proposition 2.3. *If $v \in \tilde{S_n}$ then $D_R(v) = \{s_i \in S \mid v(i) > v(i+1)\}$*

Proof. By the previous proposition, we have that

$$D_R(v) = \{s_i \in S \mid \widetilde{\text{inv}}(vs_i) < \widetilde{\text{inv}}(v)\}.$$

The rest follows from $(*)$, the recursion formula for inversion. □

Proposition 2.4. *$(\tilde{S_n}, S)$ with $S = \{s_0, \ldots, s_{n-1}\}$ is a Coxeter system.*

Proof. This is very similar to the case of the symmetric group. □

Proposition 2.5. *For $0 \leq i \leq n-1$, let $J = S \backslash \{s_i\}$. Then:*

(1) $(\tilde{S_n})_J = Stab([i+1, n+i])$

(2) $(\tilde{S_n})^J = \{v \in \tilde{S_n} \mid v(1) < v(2) < \cdots < v(i), v(i+1) < \cdots < v(n+1)\}$

Proof.
(1) Obvious
(2) Recall $(\tilde{S_n})^J = \{v \in \tilde{S_n} \mid vs > v \ \forall s \in J\}$ by definition. Then, by applying the recursion formula for inversion $(*)$, we have our result.

□

3. Minimal representatives u^J in $(\tilde{S_n})^J, J = S\backslash s_i$

Let u^J be the minimal coset representative of $(\tilde{S_n})^J$ for $J = S\backslash s_i$. By Proposition 2.5, in complete notation u^J is obtained from u by rearranging the entries $\{u(i+1+kn), \dots, u(i+n+kn)\}$ in increasing order $\forall k \in \mathbb{Z}$.

Example 3.1. Let $n = 5$.

For $u = [-3, 6, 3, -5, 14]$, notice first that this satisfies the two conditions for affine permutations stated in the beginning of these notes.

Define our set $J = \{s_0, s_1, s_2, s_4\}$ (s_3 removed), then we can write u as:

$$u = \dots \mid -3\ 6\ 3 -5\ 14 \mid 2\ 11\ 8\ 0\ 19 \mid \dots$$

We choose $-5\ 2\ 8\ 11\ 14$ in increasing order. Then we can write:

$$u^J = [3, 6, 9, -5, 2]$$

Where we obtain the first 3 entries in window notation by subtracting -5 of the last 3 elements in $-5\ 2\ 8\ 11\ 14$.

Definition 3.2. The elements in $(\tilde{S_n})^J$ for $J = \{s_1, \dots, s_{n-1}\}$ are called the Grassmannian elements.

Remark 3.3. By a lemma we proved previously (in the section about parabolic subgroups), $u \in \tilde{S_n}$ is Grassmannian:

$\iff$ no reduced expression for u ends in letters in J

$\iff$ every reduced expression for u ends in s_0

$\iff$ in window notation, $[u(1), \dots, u(n)]$, all entries are increasing.

4. Reflections for $\tilde{S_n}$

For $a, b \in \mathbb{Z}$, with $a \not\equiv b \mod n$, then define

$$t_{a,b} := \prod_{r\in\mathbb{Z}} (a + rn, b + rn)$$

Note: $s_i = t_{i,i+1}$ for $0 \le i < n$.

Proposition 4.1. *The set of reflection of $\tilde{S_n}$ is:*

$$\{t_{i,j+kn} \mid 1 \le i < j \le n, k \in \mathbb{Z}\}.$$

Proof. Let $u \in \tilde{S_n}$, $0 \le i < n$, then we have that:

$$us_iu^{-1} = \prod_{r\in\mathbb{Z}} (u(i) + rn, u(i+1) + rn)$$

Since u is any element in $\tilde{S_n}$, $u(i)$ and $u(i+1)$ can be any two elements of $\mathbb{Z}$ not congruent mod n. $\square$