MATH 280 WINTER 2009
LECTURE 3: PERMUTATION REPRESENTATION

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Note: These lecture notes follow Bjoerner and Brenti's book.

Let (W, S) be a Coxeter system.

Definition 1. S^* is the free monoid generated by S, that is, these are words in S with concatenation as product.

Define an equivalence relation $\equiv$ on S^* by allowing the insertion or deletion of any word of the form $(ss')^{m(s,s')}$ for all $s, s' \in S^2_{fin}$. As groups, $S^*/\equiv \cong W$.

Definition 2. Let $T = \{wsw^{-1} \mid s \in S, w \in W\}$ called the set of reflections.

An easy check shows these really do look like reflection: $(wsw^{-1})(wsw^{-1}) = e$. So this shows for any $t \in T, t^2 = e$, and we also have $S \subset T$. Call $s \in S \subset T$ a simple reflection.

Fix a word $s_1 s_2 \dots s_k \in S^*$. Define for all $1 \leq i \leq k$

$$t_i = s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1.$$

Define $\hat{T}(s_1 \dots s_k) = (t_1, t_2, \dots, t_k)$.

Example 3. $\hat{T}(1232) = (1, 121, 12321, 1232321)$.

Note that we can write

$$t_i = (s_1 \dots s_{i-1}) s_i (s_1 \dots s_{i-1})^{-1} \in T.$$

Observe that we have

$$t_i s_1 \dots s_k = s_1 \dots \hat{s}_i \dots s_k$$

where the hat means that that term is omitted. We also have

$$s_1 s_2 \dots s_i = t_i t_{i-1} \dots t_1.$$

Lemma 4. *Let $w = s_1 \dots s_k \in W$ with k minimal. Then $t_i \neq t_j$ for all $i \neq j$.*

Proof. By contradiction. Suppose $t_i = t_j$ for some $i < j$. We may write

$$\begin{aligned} w &= t_i t_j s_1 \dots s_k \\ &= s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k \end{aligned}$$

which contradicts the minimality of k. □

Definition 5. For $s_1 \dots s_k \in S^*$, $t \in T$, define $n(s_1 \dots s_k; t)$ =the number of times t appears in $\hat{T}(s_1 \dots s_k)$. Also define for $s \in S, t \in T$

$$\eta(s; t) = \begin{cases} 1 & \text{if } s = t \\ -1 & \text{if } s \neq t \end{cases}.$$

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Lemma 6.

$$(-1)^{n(s_1\dots s_k;t)} = \prod_{i=1}^{k} \eta(s_i; s_{i-1}\dots s_1 t s_1 \dots s_{i-1})$$

Proof. Follows from the definitions since t appears in $\hat{T}(s_1 \dots s_k)$ if $s_{i-1}\dots s_1 t s_1 \dots s_{i-1} = s_i$. □

Definition 7. Let $S(R)$ = group of permutations of R where $R = T \times \{\pm 1\}$.

Definition 8. Define $\pi_s \colon R \to R$ for $s \in S$ by $(t, \epsilon) \mapsto (sts, \epsilon\eta(s;t))$.

Lemma 9. *$\pi_s \in S(R)$.*

Proof. To obtain the result, we will show $\pi_s^2 = e$.

$$\begin{aligned}
\pi_s^2(t,\epsilon) &= \pi_s(sts, \epsilon\eta(s;t)) \\
&= (ststs s, \epsilon\eta(s;t)\eta(s, sts)) \\
&= (t, \epsilon).
\end{aligned}$$

□

Theorem 10. (i) *$s \mapsto \pi_s$ extends uniquely to an injective homomorphism $w \mapsto \pi_w$ from W to $S(R)$.*
(ii) *$\pi_t(t,\epsilon) = (t, -\epsilon)$ for all $t \in T$.*

Proof. (1) We know $\pi_s^2 = id_R$.
(2) Claim: $(\pi_s \pi_{s'})^m = id_R$ for $s, s' \in S$ and $m = m(s,s') \neq \infty$.
Proof of claim: Denote by $\underline{s}$ the word

$$\underline{s} = s_1 \dots s_{2m} = s'ss's\dots s's \qquad 2m \text{ factors}$$

and write

$$t_i = s_1 \dots s_i \dots s_1 = (s's)^{i-1}s' \qquad \text{for } 1 \leq i \leq 2m$$

then we have the following implicatons:

$$\begin{aligned}
(s's)^m = e &\Rightarrow t_{m+i} = t_i \qquad \text{for } 1 \leq i \leq m \\
&\Rightarrow n(\underline{s};t) = \text{the number of times } t = t_i, 1 \leq i \leq 2m \\
&\Rightarrow n(\underline{s};t) = \text{even for all } t \in T.
\end{aligned}$$

Let

$$(t', \epsilon') = (\pi_s \pi_{s'})^m (t, \epsilon) = \pi_{s_{2m}} \dots \pi_{s_1}(t, \epsilon)$$

then we have

$$t' = s_{2m} \dots s_1 t s_1 \dots s_{2m} = t$$

and

$$\begin{aligned}
\epsilon' &= \epsilon \prod_{i=1}^{2m} \eta(s_i; s_{i-1}\dots s_1 t s_1 \dots s_{i-1}) \\
&= \epsilon(-1)^{n(\underline{s},t)} = \epsilon
\end{aligned}$$

which finishes the proof of the claim.

(3) Let $w = s_k \dots s_1$. Then

$$\begin{aligned} \pi_w &= \pi_{s_k} \dots \pi_{s_1}(t, \epsilon) \\ &= (s_k \dots s_1 t s_1 \dots s_k, \epsilon \prod_{i=1}^{k} \eta(s_i; s_{i-1} \dots s_1 t s_1 \dots s_{i-1}) \\ &= (wtw^{-1}, \epsilon(-1)^{n(s_1 \dots s_k, t)}) \end{aligned}$$

which implies $s \mapsto \pi_s$ extends to a homomorphism $w \mapsto \pi_w$ from W to $S(W)$.

Remark 11. Since $w \mapsto \pi_w$ is well defined, we can conclude that if $s_1 \cdots s_p$ and $s'_1 \cdots s'_q$ are two expressions of the same element $w \in W$, then $(-1)^{n(s_1 \dots s_p, t)} = (-1)^{n(s'_1 \dots s'_q, t)}$. Thus we can extend the definiton of $\eta : W \times T \to \{1, -1\}$ by $\eta(w, t) = (-1)^{n(s_1 \dots s_k, t)}$ where $s_1 \dots s_k$ is an arbitrary expression of w.

(4) Claim: $w \mapsto \pi_w$ is injective.

Proof of Claim: Suppose $w \neq e$. Choose a word $w = s_k \dots s_1$ with k minimal. Recall that $\hat{T}(s_1 \dots s_k) = (t_1, \dots, t_k)$ and by a previous lemma, $t_i \neq t_j$ when $i \neq j$. Since $n(s_1 \dots s_k, t_i) = 1$ we have $\pi_w(t_i, \epsilon) = (wt_iw^{-1}, -\epsilon)$. This shows that $\pi_w \neq id$, so the map is injective.

This finishes the proof of (i).

For the proof of (ii), proceed by induction on p: $t = s_1 \dots s_p \dots s_1$ for $s_i \in S$. For $p = 1$ we see $\pi_s(s, \epsilon) = (sss, \epsilon\eta(s, s)) = (s, -\epsilon)$. Assume the result for $p - 1$. Now consider the following, applying the induction hypothesis appropriately,

$$\begin{aligned} \pi_t &= \pi_{s_1} \dots \pi_{s_p} \dots \pi_{s_1}(t, \epsilon) \\ &= \pi_{s_1} \dots \pi_{s_p} \dots \pi_{s_2}(s_1 t s_1, \epsilon\eta(s_1, t)) \\ &= \pi_{s_1} \dots \pi_{s_p} \dots \pi_{s_2}(s_2 \dots s_p \dots s_2, -\epsilon) \\ &= \pi_{s_1}(s_2 \dots s_p \dots s_2, \epsilon) \\ &= (t, -\epsilon) \end{aligned}$$

which finishes the proof of (ii). □