

## Homework 2

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**Problem 1.** Using the symmetry of the RSK algorithm, show the following:

- (1) A permutation  $\pi$  is an involution if and only if  $P(\pi) = Q(\pi)$ , where  $(P(\pi), Q(\pi))$  correspond to  $\pi$  under the RSK algorithm.
- (2) The number of involutions of  $S_n$  is  $\sum_{\lambda \vdash n} f^\lambda$ .
- (3) The number of fixed points in an involution  $\pi$  is the number of columns of odd length in  $P(\pi)$ .
- (4) We have

$$1 \cdot 3 \cdot 5 \cdots (2n - 1) = \sum_{\substack{\lambda \vdash 2n \\ \lambda^t \text{ even}}} f^\lambda,$$

where  $\lambda^t$  even means that every part in  $\lambda^t$  is even.

- (5) There is a bijection  $M \longleftrightarrow T$  between symmetric  $\mathbb{N}$ -matrices of finite support and semistandard Young tableaux such that the trace of  $M$  is the number of columns of odd length of  $T$ .
- (6) The following equations hold

$$\sum_{\lambda} s_{\lambda} = \prod_i \frac{1}{1 - x_i} \prod_{i < j} \frac{1}{1 - x_i x_j}$$

$$\sum_{\lambda^t \text{ even}} s_{\lambda} = \prod_{i < j} \frac{1}{1 - x_i x_j}.$$

**Problem 2.** Suppose  $\pi = x_1 x_2 \dots x_n \in S_n$  is a permutation in one-line notation such that  $P = P(\pi)$  has rectangular shape. Let the complement of  $\pi$  be

$$\pi^c = y_1 y_2 \dots y_n$$

where  $y_i = n + 1 - x_i$  for all  $i$ . Also define the complement of a rectangular standard tableau  $P$  with  $n$  entries to be the array obtained by replacing  $P_{ij}$  with  $n + 1 - P_{ij}$  for all  $(i, j)$  and then rotating the result by  $180^\circ$ . Show that

$$P(\pi^c) = (P^c)^t.$$

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**Problem 3.** For any symmetric polynomial  $f$ , let  $f^\perp$  be the operator adjoint to multiplication by  $f$  with respect to the Hall inner product, that is,  $\langle f^\perp g, h \rangle = \langle g, fh \rangle$  for all  $g, h \in \Lambda$ .

- (1) Find a formula for  $h_k^\perp m_\lambda$ , expressed again in terms of monomial symmetric functions  $m_\mu$ .
- (2) Show that the basis of monomial symmetric functions is uniquely characterized by the formula from the previous part.

**Problem 4.** A plane partition of  $n$  is a sequence of ordinary partitions  $\lambda = (\lambda^{(1)} \supseteq \dots \supseteq \lambda^{(k)})$  of total size  $\sum_i |\lambda^{(i)}| = n$ , weakly decreasing in the sense that the diagram of each  $\lambda^{(i)}$  is contained in that of the preceding one. The diagram of a plane partition is the three-dimensional array in  $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$  whose  $i$ -th horizontal layer is the diagram of  $\lambda^{(i)}$ .

Find a bijection between plane partitions whose diagram fits inside a  $k \times \ell \times m$  box and semi-standard Young tableaux of shape  $(k^\ell)$  with entries in  $[\ell + m]$ .