MAT 246 University of California Winter 2009

Homework 4

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Problem 1.

(1) Recall from class that

$$h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda,$$

where $z_\lambda = \prod_i i^{m_i} m_i!$ for $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$. Show that this is equivalent to Newton's determinant formula

$$h_n = \frac{1}{n!} \det \begin{pmatrix} p_1 & -1 & 0 & \ldots & 0 \\ p_2 & p_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & \cdot & \ldots & -(n-1) \\ p_n & p_{n-1} & \cdot & \ldots & p_1 \end{pmatrix}$$

(2) Show that e_n is given by the same determinant without the minus signs.

Problem 2. Prove the identity

$$s_{(n-1,n-2,\ldots,1)}(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_i + x_j).$$

Problem 3. Permutation $\pi = x_1 x_2 \cdots x_n$ has a descent at index i if $x_i > x_{i+1}$. The corresponding descent set is

$$\mathrm{Des}\ \pi = \{i \mid i \text{ is a descent of } \pi\}.$$

Similarly, i is a descent for standard tableau T if $i+1$ is in a lower row than i in T (in English notation). The descent set of T is

$$\mathrm{Des}\ T = \{i \mid i \text{ is a descent of } T\}.$$

(1) Show that if, by Robinson-Schensted, $Q(\pi) = Q$, then $\mathrm{Des}\ \pi = \mathrm{Des}\ Q$.
(2) Let $\lambda \vdash n$, $S = \{n_1 < n_2 < \cdots < n_k\} \subset \{1, 2, \ldots, n-1\}$, and $\mu = (n_1, n_2 - n_1, \ldots, n - n_k)$. Then

$$|\{\pi \in S_n \mid \mathrm{shape}(Q(\pi)) = \lambda, \mathrm{Des}\ \pi \subset S\}| = f^\lambda K_{\lambda\mu}.$$

Problem 4. Let T be a standard Young tableau of shape λ, and let b be its corner box occupied by entry 1. Define

$$\Delta(T) = \mathrm{jdt}_b(\tilde{T})$$

where $\tilde{T}$ is the skew standard tableau of shape $\lambda/(1)$ obtained from T by removing the box b and subsequently decreasing all the remaining entries by one. The evacuation tableau $\mathrm{evac}(T)$ is the standard tableau of shape λ encoded by the sequence of shapes of

$$\emptyset, \Delta^{n-1}(T), \Delta^{n-2}(T), \ldots, \Delta^2(T), \Delta(T), T.$$

For a permuation $\pi = x_1 x_2 \cdots x_n \in S_n$ define

$$w^{\#} = (n+1-x_n)\ (n+1-x_{n-1}) \cdots (n+1-x_1).$$

(1) Show: If w is mapped to (P, Q) under RSK, then $w^{\#}$ is mapped to $(\mathrm{evac}(P), \mathrm{evac}(Q))$ under RSK.
(2) Show that evac is an involution.