MAT 150B University of California Winter 2014

Homework 1

due Friday January 17, 2014 in class

1. (cf. Artin 8.1.1)
(a) Prove that every real square matrix is the sum of a symmetric matrix and a skew-symmetric matrix ($A^t = -A$) in exactly one way.

(b) Let $\langle\, , \rangle$ be a bilinear form on a real vector space V. Show that there is a symmetric form $(\, ,)$ and a skew-symmetric form $[\, ,]$ so that $\langle\, , \rangle = (\, ,) + [\, ,]$.

2. Let $\langle\, , \rangle$ be a symmetric bilinear form on a vector space V over a field F. The function $q\colon V \to F$ defined by $q(v) = \langle v, v\rangle$ is called the *quadratic form* associated to the bilinear form. Show how to recover the bilinear form from q (if the characteristic of the field F is not 2) by expanding $q(v + w)$.

3. (cf. Artin 8.4.3) A matrix B is called *positive semidefinite* if $X^tBX \geq 0$ for all $X \in \mathbb{R}^n$. Prove that $B = A^tA$ is positive semidefinite for any $m \times n$ real matrix A.

4. (cf. Artin 8.4.7) Apply the Gram-Schmidt procedure to the basis $(1, 1, 0)^t$, $(1, 0, 1)^t$, $(0, 1, 1)^t$, when the form is dot product.

5. Let A be the matrix of a symmetric bilinear form $\langle\, , \rangle$ with respect to some basis. Prove or disprove: The eigenvalues of A are independent of the basis.

6. Prove that the only real matrix which is orthogonal, symmetric, and positive definite is the identity.

7. (cf. Artin 8.4.12) Let $V = \mathbb{R}^{2\times 2}$ be the vector space of real 2×2 matrices.
(a) Determine the matrix of the bilinear form $\langle A, B\rangle = \text{trace}(AB)$ on V with respect to the standard basis $\{e_{ij}\}$.
(b) Determine the signature of this form.
(c) Find an orthogonal basis for this form.
(d) Determine the signature of the form on the subspace of V of matrices with trace zero.