Homework 1

due January 20

Problem 1. Show that the dominance partial order on partitions of n satisfies

$$\lambda \trianglelefteq \mu \quad \Longleftrightarrow \quad \lambda^t \trianglerighteq \mu^t,$$

where the t denotes the transpose of the partition.

Problem 2. For $1 \le i < j \le n$, define the raising operator R_{ij} on $\mathbb{Z}^n$ by

$$R_{ij}(\nu_1, \dots, \nu_n) = (\nu_1, \dots, \nu_i + 1, \dots, \nu_j - 1, \dots, \nu_n).$$

(1) Show that the dominance order $\trianglelefteq$ is the transitive closure of the relation on partitions $\lambda \to \mu$ if $\mu = R_{ij}\lambda$ for some $i < j$.
(2) Show that μ covers λ if and only if $\mu = R_{ij}\lambda$, where i, j satisfy the following condition: either $j = i + 1$ or $\lambda_i = \lambda_j$ (or both).
(3) Find the smallest n such that the dominance order on partitions of n is not a total ordering, and draw its Hasse diagram.

Problem 3. Let h_i be the complete homogeneous symmetric functions. Show that $u_i \in \Lambda$ satisfying $u_0 = 1$ and

$$\sum_{i=0}^{n} (-1)^i u_i h_{n-i} = 0 \qquad \text{for all } n \ge 1$$

are uniquely determined.

Problem 4. Let $w \in S_n$ be an element of the symmetric group of cycle type λ. Give a direct bijective proof that the number of elements $v \in S_n$ commuting with w is equal to

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$$

where $m_i = m_i(\lambda)$ is the number of parts of λ of size i.

Problem 5. Show that

$$\prod_{\lambda \vdash n} \prod_{i \ge 1} m_i(\lambda)! = \prod_{\lambda \vdash n} \prod_{i \ge 1} i^{m_i(\lambda)}.$$