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White-Noise and Geometrical Optics Limits of Wigner-Moyal Equation for Wave Beams in Turbulent Media*

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Abstract: Starting with the Wigner distribution formulation for beam wave propagation in Hölder continuous non-Gaussian random refractive index fields we show that the wave beam regime naturally leads to the white-noise scaling limit and converges to a Gaussian white-noise model which is characterized by the martingale problem associated to a stochastic differential-integral equation of the Itô type. In the simultaneous geometrical optics the convergence to the Gaussian white-noise model for the Liouville equation is also established if the ultraviolet cutoff or the Fresnel number vanishes sufficiently slowly. The advantage of the Gaussian white-noise model is that its n-point correlation functions are governed by closed form equations.

1. Introduction

Laser beam propagation in the turbulent atmosphere is governed by the classical wave equation with a randomly inhomogeneous refractive index field

$$n(z, \mathbf{x}) = \bar{n}(1 + \tilde{n}(z, \mathbf{x})), \quad (z, \mathbf{x}) \in \mathbb{R}^3,$$

where $\bar{n}$ is the mean and $\tilde{n}(\mathbf{x})$ is the fluctuation of the refractive index field. We seek the solution of the form $E(t, z, \mathbf{x}) = \Psi(z, \mathbf{x}) \exp[i\bar{n}(kz - wt)] + \text{c.c.}$, where E is the (scalar) electric field, k and $w = kc_0/\bar{n}$ are the carrier wavenumber and frequency, respectively, with c_0 being the wave speed in vacuum. Here and below z and $\mathbf{x}$ denote the variables in the longitudinal and transverse directions of the wave beam, respectively.

In the paraxial approximation [24], the modulation Ψ is approximated by the solution of the parabolic wave equation which after nondimensionalization with respect to some

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reference lengths L_z and L_x in the longitudinal and transverse directions, respectively, has this form

$$i\tilde{k}\frac{\partial\Psi}{\partial z}+\frac{\gamma}{2}\Delta\Psi+\tilde{k}^2k_0L_z\tilde{n}(zL_z,\mathbf{x}L_x)\Psi=0,$$
$$\Psi(0,\mathbf{x})=\Psi_0(\mathbf{x})\in L^2(\mathbb{R}^d),\quad d=2 \tag{1}$$

where $\tilde{k}=k/k_0$ is the normalized wavenumber with respect to the central wavenumber k_0 and γ is the Fresnel number

$$\gamma=\frac{L_z}{k_0L_x^2}.$$

A widely used model for the fluctuating refractive index field $\tilde{n}$ is a spatially homogeneous random field (usually assumed to be Gaussian) with the spatial structure function

$$D_n(|\vec{\mathbf{x}}|)=\mathbb{E}[\tilde{n}(\vec{\mathbf{x}}+\cdot)-\tilde{n}(\cdot)]^2=C_n^2|\vec{\mathbf{x}}|^{2/3},\quad |\vec{\mathbf{x}}|\in(\ell_0,L_0),$$
$$\vec{\mathbf{x}}=(z,\mathbf{x})\in\mathbb{R}^{d+1},\quad d=2,$$

where ℓ_0 and L_0 are the inner and outer scales, respectively. Here and below $\mathbb{E}$ stands for ensemble average.

The refractive index structure function has a spectral representation

$$D_n(|\vec{\mathbf{x}}|)=8\pi\int_0^\infty\Phi_n(|\vec{\mathbf{k}}|)\left[1-\frac{\sin(|\vec{\mathbf{k}}||\vec{\mathbf{x}}|)}{|\vec{\mathbf{k}}||\vec{\mathbf{x}}|}\right]|\vec{\mathbf{k}}|^2d|\vec{\mathbf{k}}|,\quad \vec{\mathbf{k}}\in\mathbb{R}^{d+1} \tag{2}$$

with the Kolmogorov spectral density

$$\Phi_n(|\vec{\mathbf{k}}|)=0.033C_n^2|\vec{\mathbf{k}}|^{-11/3},\quad |\vec{\mathbf{k}}|\in(\ell_0,L_0). \tag{3}$$

Here the structure parameter C_n^2 depends in general on the temperature gradient on the scales larger than L_0. See, e.g., [21, 16, 5] for more sophisticated models of turbulent refractive index fields.

In this paper we will consider a general class of spectral density parametrized by $H\in(0,1)$ and satisfying the upper bound

$$\Phi(\vec{\mathbf{k}})\le K(L_0^{-2}+|\vec{\mathbf{k}}|^2)^{-H-1/2-d/2}\left(1+\ell_0^{-2}|\mathbf{k}|^2\right)^{-2},\quad \vec{\mathbf{k}}=(\xi,\mathbf{k})\in\mathbb{R}^{d+1},\quad d=2 \tag{4}$$

for some positive constant $K<\infty$. L_0 and ℓ_0 in (4) are the infrared and ultraviolet cutoffs. The ultraviolet cutoff is physically due to dissipation on the small scales which normally results in a Gaussian decay factor [21]. We are particularly interested in the regime where the ratio L_0/ℓ_0 is large as in the high Reynolds number turbulent atmosphere.

Let us introduce the non-dimensional parameters that are pertinent to our scaling:

$$\varepsilon=\sqrt{\frac{L_x}{L_z}},\quad \eta=\frac{L_x}{L_0},\quad \rho=\frac{L_x}{\ell_0}.$$

In terms of the parameters and the power-law spectrum in (4) we rewrite (1) as

$$i\tilde{k}\frac{\partial\Psi^\varepsilon}{\partial z}+\frac{\gamma}{2}\Delta\Psi^\varepsilon+\frac{\tilde{k}^2}{\gamma}\frac{\mu}{\varepsilon}V(\frac{z}{\varepsilon^2},\mathbf{x})\Psi^\varepsilon=0,\quad \Psi^\varepsilon(0,\mathbf{x})=\Psi_0(\mathbf{x}) \tag{5}$$

with

$$\mu = \frac{\sigma L_x^H}{\varepsilon^3}, \tag{6}$$

where σ is the standard variation of the homogeneous field $\tilde{n}(z, \mathbf{x})$ and V is the normalized refractive index field with a spectral density satisfying the upper bound

$$\Phi_{\eta,\rho}(\vec{\mathbf{k}}) \leq K(\eta^2 + |\vec{\mathbf{k}}|^2)^{-H-1/2-d/2} \left(1 + \rho^{-2}|\mathbf{k}|^2\right)^{-2},$$
$$\vec{\mathbf{k}} \in \mathbb{R}^{d+1}, \quad H \in (0, 1) \tag{7}$$

for some positive constant K.

The generalized von Kármán spectral density [10, 21]

$$\Phi_{vk}(\vec{\mathbf{k}}) = 2^{H-1}\Gamma(H + \frac{d+1}{2})\eta^{2H}\pi^{-(d+1)/2}(\eta^2 + |\vec{\mathbf{k}}|^2)^{-H-1/2-d/2} \tag{8}$$

corresponds to the isotropic covariance function

$$B_{vk}(\vec{\mathbf{x}}) = \mathbb{E}\left[V(\vec{\mathbf{x}} + \cdot)V(\cdot)\right] = |\eta\vec{\mathbf{x}}|^H K_H(\eta|\vec{\mathbf{x}}|), \quad \vec{\mathbf{x}} = (z, \mathbf{x}) \in \mathbb{R}^{d+1},$$

where K_H is a Bessel function of the third kind given by

$$K_H(z) = \int_0^\infty \exp\left[-z\frac{e^t + e^{-t}}{2}\right]\frac{e^{Ht} + e^{-Ht}}{2}dt.$$

For $H = 1/2$ we have the exponential covariance function $B_{vk}(\vec{\mathbf{x}}) = \exp[-\eta|\vec{\mathbf{x}}|]$. The additional ultraviolet cutoff imposed in the upper bound (7) would then give rises to the covariance function

$$B(\vec{\mathbf{x}}) = G \star B_{vk}(\vec{\mathbf{x}}),$$

where G is the inverse Fourier transform of the cutoffs.

For high Reynolds number one has $L_0/\ell_0 = \rho/\eta \gg 1$ and thus a wide range of scales in the power spectrum (7). Note that in the worst case scenario the refractive index field loses spatial differentiability as $\rho \to \infty$ and homogeneity as $\eta \to 0$. The Gaussian field with the von Kármán spectral density (8) has H as the upper limit of the Hölder exponent of the sample field. The Kolmogorov spectrum has the exponent $H = 1/3$. Since our result does not depend on d we hereafter take it to be any positive integer.

Although we do not assume isotropic spectral densities, the spectral density always satisfies the basic symmetry:

$$\Phi_{(\eta,\rho)}(\xi, \mathbf{k}) = \Phi_{(\eta,\rho)}(-\xi, \mathbf{k}) = \Phi_{(\eta,\rho)}(\xi, -\mathbf{k}), \quad \forall(\xi, \mathbf{k}) \in \mathbb{R}^{d+1}. \tag{9}$$

In other words, the spectral density is invariant under change of sign in any component of the argument because the underlying stochastic process is real-valued.

We also assume that $V_z(\mathbf{x}) \equiv V(z, \mathbf{x})$ is a centered, square-integrable, z-stationary and $\mathbf{x}$-homogeneous process with the (partial) spectral representation

$$V_z(\mathbf{x}) = \int \exp(i\mathbf{p} \cdot \mathbf{x})\widehat{V}_z(d\mathbf{p}), \tag{10}$$

where the process $\widehat{V}_z(d\mathbf{p})$ is the $z-$stationary orthogonal spectral measure satisfying

$$\mathbb{E}\left[\hat{V}_z(d\mathbf{p})\hat{V}_z(d\mathbf{q})\right] = \delta(\mathbf{p}+\mathbf{q})\left[\int \Phi(w,\mathbf{p})dw\right]d\mathbf{p}d\mathbf{q}. \tag{11}$$

We do *not* assume the Gaussian property but instead a sub-Gaussian property (see Sect. 3.1 for precise statements).

If the observation scales L_z and L_x are the longitudinal and transverse scales, respectively, of the wave beam then $\varepsilon \ll 1$ corresponds to a long, narrow wave beam. The white-noise scaling then corresponds to $\varepsilon \to 0$ with a fixed μ. For convenience we set $\mu = 1$. The white-noise scaling limit $\varepsilon \to 0$ of Eq. (5) is analyzed in [3, 4, 11]. The limit $\gamma \to 0$ corresponds to the geometrical optics limit. In this paper we study the higher moments behavior in both white-noise and geometrical optics limits by considering the Wigner transform of the modulation function.

Our method is also suitable for the situation where deterministic large-scale inhomogeneities are present. One type of slowly varying, large-scale inhomogeneities is multiplicative and can be modeled by a bounded smooth deterministic function $\mu = \mu(z,\mathbf{x})$ due to variability of any one of the three factors in (6) (see, e.g., [5, 2] for models with slowly varying σ). The second type is additive and can be modeled by adding a smooth background $V_0(z,\mathbf{x})$. Altogether we can treat the random refractive index field of the general type

$$V_0(z,\mathbf{x}) + \frac{\mu(z,\mathbf{x})}{\varepsilon} V\left(\frac{z}{\varepsilon^2},\mathbf{x}\right)$$

with a bounded smooth deterministic modulation $\mu(z,\mathbf{x})$ and background $V_0(z,\mathbf{x})$. We describe the results in Sect. 2.3 but omit the details of the argument for simplicity of presentation. As the small-scale turbulent fluctuations are invariably embedded in a structure determined by large-scale geophysics this generalization is necessary for practical application of the scaling limits.

1.1. Wigner distribution and Wigner-Moyal equation. The Wigner transform of Ψ^ε, called the Wigner distribution, is defined as

$$W_z^\varepsilon(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi^\varepsilon\left(z,\mathbf{x}+\frac{\gamma\mathbf{y}}{2}\right)\Psi^{\varepsilon*}\left(z,\mathbf{x}-\frac{\gamma\mathbf{y}}{2}\right)d\mathbf{y}. \tag{12}$$

One has the following bounds from (12)

$$\|W_z^\varepsilon\|_\infty \le (2\gamma\pi)^{-d}\|\Psi^\varepsilon(z,\cdot)\|_2^2, \quad \|W_z^\varepsilon\|_2 = (2\gamma\pi)^{-d/2}\|\Psi^\varepsilon(z,\cdot)\|_2^2$$

[13, 15, 20]. The Wigner distribution has many important properties. For instance, it is real and its $\mathbf{p}$-integral is the modulus square of the function ϕ,

$$\int_{\mathbb{R}^d} W^\varepsilon(\mathbf{x},\mathbf{p})d\mathbf{p} = |\Psi^\varepsilon(\mathbf{x})|^2, \tag{13}$$

so we may think of $W(\mathbf{x},\mathbf{p})$ as wave number-resolved mass density. Additionally, its $\mathbf{x}$-integral is

$$\int_{\mathbb{R}^d} W^\varepsilon(\mathbf{x},\mathbf{p})d\mathbf{x} = \left(\frac{2\pi}{\gamma}\right)^d |\widehat{\Psi}^\varepsilon|^2(\mathbf{p}/\gamma).$$

The energy flux is expressed through $W^\varepsilon(\mathbf{x},\mathbf{p})$ as

$$\frac{1}{2i}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) = \int_{\mathbb{R}^d} \mathbf{p}W^\varepsilon(\mathbf{x},\mathbf{p})d\mathbf{p} \tag{14}$$

and its second moment in $\mathbf{p}$ is

$$\int |\mathbf{p}|^2 W(\mathbf{x},\mathbf{p})d\mathbf{p} = |\nabla\Psi^\varepsilon(\mathbf{x})|^2. \tag{15}$$

In view of these properties it is tempting to think of the Wigner distribution as a phase-space probability density, which is unfortunately not the case, since it is not everywhere non-negative. Nevertheless, the Wigner distribution is a useful tool for analyzing the evolution of wave energy in the phase space. Moreover, in the recent development of time reversal of waves in which a part of the waves is received, phase-conjugated and then back-propagated toward the source the refocused wave field is given by a Wigner distribution of mixed-state type (see (25) below) [7, 23, 12].

The Wigner distribution, written as $W^\varepsilon_z(\mathbf{x},\mathbf{p}) = W^\varepsilon(z,\mathbf{x},\mathbf{p})$, satisfies an evolution equation, called the Wigner-Moyal equation,

$$\frac{\partial W^\varepsilon_z}{\partial z} + \frac{\mathbf{p}}{\tilde{k}}\cdot\nabla_\mathbf{x} W^\varepsilon_z + \frac{\tilde{k}}{\varepsilon}\mathcal{L}^\varepsilon_z W^\varepsilon_z = 0 \tag{16}$$

with the initial data

$$W_0(\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int e^{i\mathbf{k}\cdot\mathbf{y}}\Psi_0(\mathbf{x} - \frac{\gamma\mathbf{y}}{2})\Psi_0^*(\mathbf{x} + \frac{\gamma\mathbf{y}}{2})d\mathbf{y}\,, \tag{17}$$

where the operator $\mathcal{L}^\varepsilon_z$ is formally given as

$$\begin{aligned}\mathcal{L}^\varepsilon_z W^\varepsilon_z &= i\int e^{i\mathbf{q}\cdot\mathbf{x}}\gamma^{-1}\left[W^\varepsilon_z(\mathbf{x},\mathbf{p}+\gamma\mathbf{q}/2) - W^\varepsilon_z(\mathbf{x},\mathbf{p}-\gamma\mathbf{q}/2)\right]\widehat{V}(\frac{z}{\varepsilon^2},d\mathbf{q})\\ &= 2\gamma^{-1}\int W^\varepsilon_z(\mathbf{x},\gamma\mathbf{q}/2)\mathrm{Im}\left[e^{-i2\gamma^{-1}\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{x}}\widehat{V}(\frac{z}{\varepsilon^2},d\mathbf{q})\right].\end{aligned} \tag{18}$$

We will use the following definition of the Fourier transform and inversion:

$$\mathcal{F}f(\mathbf{p}) = \frac{1}{(2\pi)^d}\int e^{-i\mathbf{x}\cdot\mathbf{p}}f(\mathbf{x})d\mathbf{x},$$
$$\mathcal{F}^{-1}g(\mathbf{x}) = \int e^{i\mathbf{p}\cdot\mathbf{x}}g(\mathbf{p})d\mathbf{p}.$$

When making a *partial* (inverse) Fourier transform on a phase-space function we will write $\mathcal{F}_1$ (resp. $\mathcal{F}_1^{-1}$) and $\mathcal{F}_2$ (resp. $\mathcal{F}_2^{-1}$) to denote the (resp. inverse) transform w.r.t. $\mathbf{x}$ and $\mathbf{p}$ respectively.

A useful way of analyzing $\mathcal{L}^\varepsilon_z W^\varepsilon_z$ as formally given in (18) is to look at its partial inverse Fourier transform $\mathcal{F}_2^{-1}\mathcal{L}^\varepsilon_z W^\varepsilon_z(\mathbf{x},\mathbf{y})$ acting on

$$\mathcal{F}_2^{-1}W^\varepsilon_z(\mathbf{x},\mathbf{y}) \equiv \int e^{i\mathbf{p}\cdot\mathbf{y}}W^\varepsilon_z(\mathbf{x},\mathbf{p})\,d\mathbf{p} = \Psi^\varepsilon(\mathbf{x}+\gamma\mathbf{y}/2)\Psi^{\varepsilon*}(\mathbf{x}-\gamma\mathbf{y}/2)$$

in the following completely local manner:

$$\mathcal{F}_2^{-1}\mathcal{L}^\varepsilon_z W^\varepsilon_z(\mathbf{x},\mathbf{y}) = -i\gamma^{-1}\delta_\gamma V^\varepsilon_z(\mathbf{x},\mathbf{y})\mathcal{F}_2^{-1}W^\varepsilon_z(\mathbf{x},\mathbf{y}), \tag{19}$$

where

$$\delta_\gamma V_z^\varepsilon(\mathbf{x},\mathbf{y}) \equiv V_z^\varepsilon(\mathbf{x}+\gamma\mathbf{y}/2) - V_z^\varepsilon(\mathbf{x}-\gamma\mathbf{y}/2), \tag{20}$$

$$V_z^\varepsilon(\mathbf{x}) = V_{z/\varepsilon^2}(\mathbf{x}). \tag{21}$$

Hereby we define for every realization of V_z^ε the operator $\mathcal{L}_z^\varepsilon$ to act on a phase-space test function θ as

$$\mathcal{L}_z^\varepsilon \theta(\mathbf{x},\mathbf{p}) \equiv -i\gamma^{-1}\mathcal{F}_2\left[\delta_\gamma V_z^\varepsilon(\mathbf{x},\mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\right] \tag{22}$$

with the difference operator δ_γ given by (20) for any test function $\theta \in \mathcal{S}$, where

$$\mathcal{S} = \left\{\theta(\mathbf{x},\mathbf{p}) \in L^2(\mathbb{R}^{2d});\, \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \in C_c^\infty(\mathbb{R}^{2d})\right\}.$$

We note that $\mathcal{L}_z^\varepsilon$ is skew-symmetric and real (i.e. mapping real-valued functions to real-valued functions). In this paper we consider the weak formulation of the Wigner-Moyal equation: To find $W_z^\varepsilon \in D([0,\infty); L^2(\mathbb{R}^{2d}))$ such that $\|W_z^\varepsilon\|_2 \le \|W_0\|_2, \forall z > 0$, and

$$\left\langle W_z^\varepsilon,\theta\right\rangle - \langle W_0,\theta\rangle = \tilde{k}^{-1}\int_0^z \left\langle W_s^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle ds + \frac{\tilde{k}}{\varepsilon}\int_0^z \left\langle W_s^\varepsilon, \mathcal{L}_s^\varepsilon\theta\right\rangle ds, \quad \forall\theta\in\mathcal{S}. \tag{23}$$

Remark 1. Since Eq. (23) is linear , the existence of weak solutions can be established straightforwardly by the weak-$\star$ compactness argument. Let us briefly comment on this. Without loss of generality we set $\varepsilon = 1$. First, we introduce truncation $N < \infty$,

$$V_N(z,\mathbf{x}) = \mathbb{I}_N V(z,\mathbf{x}),$$

where $\mathbb{I}_N$ is the characteristic function of the set $\{|V(z,x)| < N\}$. Clearly, for such bounded V_N the corresponding operator $\mathcal{L}_z^\varepsilon$ is a bounded skew-adjoint operator on $L^2(\mathbb{R}^{2d})$. Hence the corresponding Wigner-Moyal equation gives rise to a unique group of unitary maps on L^2. Let us denote the solution by $W_z^{(N)}$. Passing to the limit $N \to \infty$ by selecting a weakly convergent subsequence we obtain a L^2-weak solution for the Wigner-Moyal equation with the truncation removed if V is locally square-integrable as is assumed here. The limiting solution W_z has a L^2-norm equal to or less than that of W_0.

Moreover, from Eq. (23), it is easy to see that $\left\langle W_z^{(N)},\theta\right\rangle$ is equi-continuous on any compact subset of $z \in \mathbb{R}$. By the Arzela-Ascoli Lemma, $\langle W_z,\theta\rangle$ is z-continuous almost surely. Because $\left\langle W_z^{(N)},\theta\right\rangle$ is adapted to the filtration of V_z and the convergence is almost sure, the resulting solution W_z is adapted to the filtration of V_z.

We will not address the uniqueness of solution for the Wigner-Moyal equation (23) but we will show that as $\varepsilon \to 0$ any sequence of weak solutions to Eq. (23) converges in a suitable sense to the unique solution of a martingale problem (see Theorem 1 and 2).

1.2. Liouville equation. In the geometrical optics limit $\gamma \to 0$, if one takes the usual WKB-type initial condition

$$\Psi(0, \mathbf{x}) = A_0(\mathbf{x}) e^{iS(\mathbf{x})/\gamma},$$

then the Wigner distribution formally tends to the WKB-type distribution

$$W_0(\mathbf{x}, \mathbf{p}) = |A_0|^2 \delta(\mathbf{p} - \nabla S(\mathbf{x})) \tag{24}$$

which satisfies $\mathcal{F}_2^{-1} W_0 \in L^\infty(\mathbb{R}^{2d})$. It has been shown [6] that the primitive WKB-type distribution (24) can *not* arise from the geometrical optics limit ($\gamma \to 0$) from any *pure* state Wigner distribution as given by (17) but rather from a *mixed* state Wigner distribution of the form

$$W_0(\mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^d} \int \int e^{i\mathbf{k}\cdot\mathbf{y}} \Psi_0(\mathbf{x} - \frac{\gamma \mathbf{y}}{2}; \alpha) \Psi_0^*(\mathbf{x} + \frac{\gamma \mathbf{y}}{2}; \alpha) d\mathbf{y} dP(\alpha), \tag{25}$$

where $P(\alpha)$ is a probability distribution of a family of states $\Psi_0(\cdot, \alpha)$ parametrized by α. The mixed state Wigner distributions generally give rise to a smeared initial condition, i.e. $W_0(\mathbf{x}, \mathbf{p}) \in L^2(\mathbb{R}^{2d})$ even in the geometrical optics limit. This, instead of the WKB type, is the kind of initial conditions considered in this paper.

When acting on the test function space $\mathcal{S}$, $\mathcal{L}_z^\varepsilon$ as given by (22) has the following limit:

$$\lim_{\gamma \to 0} \mathcal{L}_z^\varepsilon \theta(\mathbf{x}, \mathbf{p}) = -\mathcal{F}_2 \left[\nabla_\mathbf{x} V_z^\varepsilon(\mathbf{x}) \cdot \left[i\mathbf{y} \mathcal{F}_2^{-1} \theta(\mathbf{x}, \mathbf{y}) \right] \right] = -\nabla_\mathbf{x} V_z(\mathbf{x}) \cdot \nabla_\mathbf{p} \theta(\mathbf{x}, \mathbf{p}) \tag{26}$$

in the L^2-sense for all $\theta \in \mathcal{S}$ and all locally square-integrable V_z. Hence the Wigner-Moyal equation (23) formally becomes in the limit $\gamma \to 0$ the Liouville equation in the weak formulation

$$\begin{aligned} \left\langle W_z^\varepsilon, \theta \right\rangle - \langle W_0, \theta \rangle = \tilde{k}^{-1} \int_0^z \left\langle W_s^\varepsilon, \mathbf{p} \cdot \nabla_\mathbf{x} \theta \right\rangle ds \\ - \frac{\tilde{k}}{\varepsilon} \int_0^z \left\langle W_s^\varepsilon, \nabla_\mathbf{x} V_s^\varepsilon \cdot \nabla_\mathbf{p} \theta \right\rangle ds, \quad \forall \theta \in \mathcal{S}. \end{aligned} \tag{27}$$

The same weak-$\star$ compactness argument as described in Remark 1 establishes the existence of L^2-weak solution of the Liouville equation except now that the operator (26) is unbounded and requires local square integrability of $\nabla V_z(\cdot)$. We will show that as $\varepsilon \to 0$ any sequence of weak solutions of the Wigner-Moyal equation with any L^2-initial condition converge as $\varepsilon, \gamma \to 0$ in a suitable sense to the unique solution of a martingale problem associated with the Gaussian white-noise model of the Liouville equation (see Theorem 2).

In addition to the limit $\varepsilon \to 0$ we shall also let $\rho \to \infty$ and $\eta \to 0$ simultaneously. We first study the case $\rho \to \infty$, but η fixed, as $\varepsilon \to 0$. This means that the Fresnel length is comparable to the outer scale. Then we study the narrow beam regime $\eta \to 0$, where the Fresnel length is in the middle of the inertial-convective subrange.

2. Formulation

2.1. Martingale formulation. The tightness result (see below) implies that for L^2 initial data the limiting measure $\mathbb{P}$ is supported in $L^2([0, z_0]; L^2(\mathbb{R}^{2d}))$. For tightness as well as identification of the limit, the following infinitesimal operator $\mathcal{A}^\varepsilon$ will play an important role. Let $V_z^\varepsilon \equiv V(z/\varepsilon^2, \cdot)$ and $z_0 < \infty$ be any positive number. Let $\mathcal{F}_z^\varepsilon$ be the σ-algebras generated by $\{V_s^\varepsilon,\ s \le t\}$ and $\mathbb{E}_z^\varepsilon$ the corresponding conditional expectation w.r.t. $\mathcal{F}_z^\varepsilon$. Let $\mathcal{M}^\varepsilon$ be the space of a measurable function adapted to $\{\mathcal{F}_z^\varepsilon, z \in \mathbb{R}\}$ such that $\sup_{z<z_0} \mathbb{E}|f_z| < \infty$. We say $f_z \in \mathcal{D}(\mathcal{A}^\varepsilon)$, the domain of $\mathcal{A}^\varepsilon$, and $\mathcal{A}^\varepsilon f_z = g_z$ if $f_z, g_z \in \mathcal{M}^\varepsilon$ and for $f_z^\delta \equiv \delta^{-1}[\mathbb{E}_z^\varepsilon f_{z+\delta} - f_z]$ we have

$$\sup_{z,\delta>0} \mathbb{E}|f_z^\delta| < \infty,$$
$$\lim_{\delta\to 0} \mathbb{E}|f_z^\delta - g_z| = 0, \quad \forall t.$$

Consider a special class of admissible functions $f_z = f(\langle W_z^\varepsilon, \theta\rangle)$, $f_z' = f'(\langle W_z^\varepsilon, \theta\rangle)$, $\forall f \in C^\infty(\mathbb{R})$. We have the following expression from (23) and the chain rule:

$$\mathcal{A}^\varepsilon f_z = f_z' \left[\frac{1}{\tilde{k}} \left\langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_{\mathbf{x}} \theta \right\rangle + \frac{\tilde{k}}{\varepsilon} \left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \theta \right\rangle \right]. \tag{28}$$

A main property of $\mathcal{A}^\varepsilon$ is that

$$f_z - \int_0^z \mathcal{A}^\varepsilon f_s ds \quad \text{is a } \mathcal{F}_z^\varepsilon\text{-martingale}, \quad \forall f \in \mathcal{D}(\mathcal{A}^\varepsilon). \tag{29}$$

Also,

$$\mathbb{E}_s^\varepsilon f_z - f_s = \int_s^z \mathbb{E}_s^\varepsilon \mathcal{A}^\varepsilon f_\tau d\tau \quad \forall s < z \quad \text{a.s.} \tag{30}$$

(see [18]). Note that the process W_z^ε is not Markovian and $\mathcal{A}^\varepsilon$ is not its generator. We denote by $\mathcal{A}$ the infinitesimal operator corresponding to the unscaled process $V_z(\cdot) = V(z, \cdot)$.

2.2. White-noise models. Now we formulate the solutions for the Gaussian white-noise model as the solutions to the corresponding martingale problem: Find the law $\mathbb{Q}$ on $\mathcal{Z} = D([0, \infty); L_w^2(\mathbb{R}^{2d}))$ such that for $\zeta \in \mathcal{Z}$ and $W_z(\omega) \equiv \zeta(z)$, $z \ge 0$ we have that $\mathbb{Q}\left(W_0(\omega) = W_0 \in L^2(\mathbb{R}^{2d})\right) = 1$ and that

$$\begin{aligned} f(\langle W_z, \theta\rangle) - \int_0^z \Big\{ & f'(\langle W_s, \theta\rangle) \left[\frac{1}{\tilde{k}} \langle W_s, \mathbf{p} \cdot \nabla_{\mathbf{x}} \theta\rangle + \tilde{k}^2 \left\langle W_s, \overline{\mathcal{Q}}_0 \theta \right\rangle \right] \\ & + \tilde{k}^2 f''(\langle W_s, \theta\rangle) \left\langle W_s, \overline{\mathcal{K}}_\theta W_s \right\rangle \Big\} ds \\ & \text{is a martingale for each } f \in C^\infty(\mathbb{R}) \end{aligned}$$

with

$$\overline{\mathcal{K}}_\theta W_s = \int \overline{\mathcal{Q}}(\theta \otimes \theta)(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) W_s(\mathbf{y}, \mathbf{q})\, d\mathbf{y} d\mathbf{q}. \tag{31}$$

Here, in the case of the white-noise model for the Wigner-Moyal equation (Theorem 1), the covariance operators $\overline{\mathcal{Q}}$, $\overline{\mathcal{Q}}_0$ are defined as

$$\overline{\mathcal{Q}}_0\theta = \int \Phi_\eta^\infty(\mathbf{q})\gamma^{-2}\left[-2\theta(\mathbf{x},\mathbf{p}) + \theta(\mathbf{x},\mathbf{p}-\gamma\mathbf{q}) + \theta(\mathbf{x},\mathbf{p}+\gamma\mathbf{q})\right]d\mathbf{q}, \tag{32}$$

$$\overline{\mathcal{Q}}(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) = \int e^{i\mathbf{q}'\cdot(\mathbf{x}-\mathbf{y})}\Phi_\eta^\infty(\mathbf{q}')\gamma^{-2}\left[\theta(\mathbf{x},\mathbf{p}-\gamma\mathbf{q}'/2) - \theta(\mathbf{x},\mathbf{p}+\gamma\mathbf{q}'/2)\right] \times\left[\theta(\mathbf{y},\mathbf{q}-\gamma\mathbf{q}'/2) - \theta(\mathbf{y},\mathbf{q}+\gamma\mathbf{q}'/2)\right]d\mathbf{q}' \tag{33}$$

and, in the case of the white-noise model for the Liouville equation (Theorem 2),

$$\overline{\mathcal{Q}}_0\theta(\mathbf{x},\mathbf{p}) = \nabla_{\mathbf{p}}\cdot\int \Phi_\eta^\rho(\mathbf{q})\mathbf{q}\otimes\mathbf{q}\,d\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p}), \tag{34}$$

$$\overline{\mathcal{Q}}(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) = \nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})\cdot\left[\int e^{i\mathbf{q}'\cdot(\mathbf{x}-\mathbf{y})}\Phi_\eta^\rho(\mathbf{q}')\mathbf{q}'\otimes\mathbf{q}'d\mathbf{q}'\right]\cdot\nabla_{\mathbf{q}}\theta(\mathbf{y},\mathbf{q}), \quad \eta\ge 0,\ \ \rho<\infty, \tag{35}$$

with the spectral density $\Phi_\eta^\infty(\mathbf{q})$ given by

$$\Phi_\eta^\infty(\mathbf{q}) = \lim_{\rho\to\infty}\Phi_\eta^\rho(\mathbf{q}) \equiv \lim_{\rho\to\infty}\Phi_{\eta,\rho}(0,\mathbf{q}), \qquad \eta\ge 0.$$

Note that the operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}}_0$ are well-defined for any test function $\theta\in\mathcal{S}$ in the former case for any $H\in(0,1)$, $\eta>0$ or $\eta=0$, $H\in(0,1/2)$, and in the latter case for $H\in(0,1), 0\eta<\rho<\infty$ or $H\in(0,1/2), 0=\eta<\rho<\infty$ or $H\in(1/2,1), 0<\eta<\rho=\infty$.

That the martingale problem as formulated with the special class of test functions is sufficient to characterize the law $\mathbb{Q}$ follows from the uniqueness result discussed in Sect. 2.4.

To see that (31)–(33) is square-integrable and well-defined for any $L^2(\mathbb{R}^{2d})$-valued process W_z, we apply $\mathcal{F}_2^{-1}$ to (31) and obtain

$$\mathcal{F}_2^{-1}\overline{\mathcal{K}}_\theta W_s(\mathbf{x},\mathbf{x}') = \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{x}')\int e^{i\mathbf{q}'\cdot(\mathbf{x}-\mathbf{y})}\Phi_\eta^\infty(\mathbf{q}')\gamma^{-2}\left[e^{i\gamma\mathbf{q}'\cdot\mathbf{x}'/2} - e^{-i\gamma\mathbf{q}'\cdot\mathbf{x}'/2}\right] \times\left[\theta(\mathbf{y},\mathbf{q}-\gamma\mathbf{q}'/2) - \theta(\mathbf{y},\mathbf{q}+\gamma\mathbf{q}'/2)\right]W_z(\mathbf{y},\mathbf{q})d\mathbf{y}d\mathbf{q}d\mathbf{q}' \tag{36}$$

$$= (2\pi)^{-d}\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{x}')\int \mathcal{F}_2^{-1}\theta(\mathbf{y},\mathbf{y}')\mathcal{F}_2^{-1}W_z(\mathbf{y},-\mathbf{y}') \times\int e^{i\mathbf{q}'\cdot(\mathbf{x}-\mathbf{y})}\Phi_\eta^\infty(\mathbf{q}')\gamma^{-2}\left[e^{i\gamma\mathbf{q}'\cdot\mathbf{x}'/2} - e^{-i\gamma\mathbf{q}'\cdot\mathbf{x}'/2}\right] \times\left[e^{i\gamma\mathbf{q}'\cdot\mathbf{y}'/2} - e^{-i\gamma\mathbf{q}'\cdot\mathbf{y}'/2}\right]\mathbf{q}'d\mathbf{y}d\mathbf{y}'. \tag{37}$$

The integral on the right side of (36) is bounded over compact sets of $(\mathbf{x},\mathbf{x}')$ because firstly $\theta\in\mathcal{S}$, $W_z\in L^2(\mathbb{R}^{2d})$, and secondly the function

$$\Phi_\eta^\infty(\mathbf{q}')\left[e^{i\gamma\mathbf{q}'\cdot\mathbf{x}'/2} - e^{-i\gamma\mathbf{q}'\cdot\mathbf{x}'/2}\right]\left[e^{i\gamma\mathbf{q}'\cdot\mathbf{y}'/2} - e^{-i\gamma\mathbf{q}'\cdot\mathbf{y}'/2}\right]$$

is integrable in $\mathbf{q}' \in \mathbb{R}^d$ and the associated integral is bounded over compact sets of $\mathbf{x}'$ for any $H \in (0, 1)$, $\eta > 0$ or $\eta = 0$, $H < 1/2$. Hence the function on the right side of (36) has a compact support and is square-integrable. Similarly, one can show that (32)–(35) is well defined for $H \in (0, 1)$, $\rho < \infty$ or $H > 1/2$, $\rho = \infty$.

In view of the martingale problem the white-noise model is an infinite-dimensional Markov process whose generator when applied to the special class of test functions f_z has the form

$$\bar{\mathcal{A}} f_z \equiv f_s' \left[\frac{1}{\tilde{k}} \langle W_z, \mathbf{p} \cdot \nabla_{\mathbf{x}} \theta \rangle + \tilde{k}^2 \bar{A}_1(W_z) \right] + \tilde{k}^2 f_z'' \bar{A}_2(W_z).$$

This Markov process W_z can also be formulated as *weak* solutions to the Itô's equation

$$dW_z = \left(\frac{-1}{\tilde{k}} \mathbf{p} \cdot \nabla_{\mathbf{x}} + \tilde{k}^2 \overline{\mathcal{Q}}_0 \right) W_z \, dz + \tilde{k} d\overline{\mathcal{B}}_z W_z, \quad W_0(\mathbf{x}) \in L^2(\mathbb{R}^{2d}) \tag{38}$$

or as the Stratonovich's equation

$$dW_z = \frac{-1}{\tilde{k}} \mathbf{p} \cdot \nabla_{\mathbf{x}} + \tilde{k} d\overline{\mathcal{B}}_z \circ W_z, \quad W_0(\mathbf{x}) \in L^2(\mathbb{R}^{2d}),$$

where $\overline{\mathcal{B}}_z$ is the operator-valued Brownian motion with the covariance operator $\overline{\mathcal{Q}}$, i.e.

$$\mathbb{E}\left[d\overline{\mathcal{B}}_z \theta(\mathbf{x}, \mathbf{p}) d\bar{\mathcal{B}}_{z'} \theta(\mathbf{y}, \mathbf{q}) \right] = \delta(z - z') \overline{\mathcal{Q}}(\theta \otimes \theta)(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) dz dz'.$$

Equation (38) should be solved in the space $D([0, \infty); L^2_w(\mathbb{R}^{2d}))$, namely, to find $W_z \in D([0, \infty); L^2_w(\mathbb{R}^{2d}))$ such that for all $\theta \in L^2(\mathbb{R}^{2d})$,

$$d \langle W_z, \theta \rangle = \left\langle W_z, \left(\frac{1}{\tilde{k}} \mathbf{p} \cdot \nabla_{\mathbf{x}} + \tilde{k}^2 \overline{\mathcal{Q}}_0 \right) \theta \right\rangle dz + \tilde{k} \left\langle W_z, d\overline{\mathcal{B}}_z \theta \right\rangle, \quad W_0(\mathbf{x}) \in L^2(\mathbb{R}^{2d}). \tag{39}$$

Our results show that the *weak* solution to (39) exists, is unique and satisfies the L^2-bound

$$\| W_z \|_2 \leq \| W_0 \|_2$$

(cf. Theorem 1, 2, Remark 1, 2 and Sect. 2.4).

In view of (33), (32), (34) and (35) we can interpret the white-noise limit $\varepsilon \to 0$ as giving rise to a white-noise-in-z potential V_z^* whose spectral density is bounded from above by

$$K^* (\eta^2 + |\mathbf{k}|^2)^{-H^* - d/2}$$

for some constant $K^* < \infty$ with the effective Hölder exponent $H_* = H + 1/2$ by observing that

$$\lim_{\varepsilon \to 0} \mathcal{L}_z^\varepsilon \theta(\mathbf{x}, \mathbf{p}) = -i \mathcal{F}_2 \left[\gamma^{-1} \delta_\gamma V_z^*(\mathbf{x}, \mathbf{y}) \mathcal{F}_2^{-1} \theta(\mathbf{x}, \mathbf{y}) \right], \quad \forall \theta \in \mathcal{S}, \tag{40}$$

$$\lim_{\varepsilon, \gamma \to 0} \mathcal{L}_z^\varepsilon \theta(\mathbf{x}, \mathbf{p}) = \nabla_{\mathbf{x}} V_z^*(\mathbf{x}) \cdot \nabla_{\mathbf{p}} \theta(\mathbf{x}, \mathbf{p}), \quad \forall \theta \in \mathcal{S} \tag{41}$$

in the mean square sense.

2.3. White-noise models with large-scale inhomogeneities. First we consider the case of deterministic, large-scale inhomogeneities of a multiplicative type which has μ, given by (6), as a bounded smooth function $\mu = \mu(z, \mathbf{x})$. The resulting limiting process can be described analogously as above except with the term Φ^∞_η replaced by

$$\Phi^\infty_\eta(\mathbf{k}) \longrightarrow \mu(z,\mathbf{x})\mu(z,\mathbf{y})\Phi^\infty_\eta(\mathbf{k}), \quad \text{in} \quad \overline{\mathcal{Q}},$$
$$\Phi^\infty_\eta(\mathbf{k}) \longrightarrow \mu^2(z,\mathbf{x})\Phi^\infty_\eta(\mathbf{k}), \quad \text{in} \quad \overline{\mathcal{Q}}_0.$$

As a consequence the operator $\overline{\mathcal{Q}}_0$ is no longer of convolution type.

Next we add a slowly varying smooth deterministic background $V_0(z, \mathbf{x})$ to the rapidly fluctuating field $\varepsilon^{-1}\mu(z, \mathbf{x})V(\varepsilon^{-2}z, \mathbf{x})$. Namely we have

$$V_0(z,\mathbf{x}) + \frac{\mu(z,\mathbf{x})}{\varepsilon} V(\frac{z}{\varepsilon^2}, \mathbf{x})$$

as the potential term in the parabolic wave equation (5).

The resulting martingale problem has an additional term

$$-\int_0^z \tilde{k}\,\langle W_s, \mathcal{L}_0\theta\rangle\, ds \tag{42}$$

in the martingale formulation where $\mathcal{L}_0\theta$ has the form

$$\begin{aligned}\mathcal{L}_0\theta(\mathbf{x},\mathbf{p}) &= i\int e^{i\mathbf{q}\cdot\mathbf{x}}\gamma^{-1}\left[\theta(\mathbf{x},\mathbf{p}+\gamma\mathbf{q}/2) - \theta(\mathbf{x},\mathbf{p}-\gamma\mathbf{q}/2)\right]\widehat{V}_0(z,d\mathbf{q})\\ &\equiv -i\gamma^{-1}\mathcal{F}_2\left[(V_0(\mathbf{x}+\gamma\mathbf{y}/2) - V_0(\mathbf{x}-\gamma\mathbf{y}/2))\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\right]\end{aligned} \tag{43}$$

for $\gamma > 0$ fixed in the limit, and the form

$$\mathcal{L}_0\theta(\mathbf{x},\mathbf{p}) = -\nabla_{\mathbf{x}}V_0(z,\mathbf{x})\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p}) \tag{44}$$

in the case of $\gamma \to 0$.

2.4. Multiple-point correlation functions of the limiting model. The martingale solutions of the limiting models are uniquely determined by their n-point correlation functions which satisfy a closed set of evolution equations.

Using the function $f(r) = r^n$ in the martingale formulation and taking expectation, we arrive after some algebra at the following equation:

$$\begin{aligned}\frac{\partial F^{(n)}}{\partial z} &= \frac{1}{\tilde{k}}\sum_{j=1}^n \mathbf{p}_j\cdot\nabla_{\mathbf{x}_j}F^{(n)} + \tilde{k}^2\sum_{j=1}^n \overline{\mathcal{Q}}_0(\mathbf{x}_j,\mathbf{p}_j)F^{(n)}\\ &\quad +\tilde{k}^2\sum_{\substack{j,k=1\\ j\neq k}}^n \overline{\mathcal{Q}}(\mathbf{x}_j,\mathbf{p}_j,\mathbf{x}_k,\mathbf{p}_k)F^{(n)}\end{aligned} \tag{45}$$

for the $n-$point correlation function

$$F^{(n)}(z,\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_n,\mathbf{p}_n) \equiv \mathbb{E}\left[W_z(\mathbf{x}_1,\mathbf{p}_1)\cdots W_z(\mathbf{x}_n,\mathbf{p}_n)\right],$$

where $\overline{\mathcal{Q}}_0(\mathbf{x}_j, \mathbf{p}_j)$ is the operator $\overline{\mathcal{Q}}_0$ acting on the variables $(\mathbf{x}_j, \mathbf{p}_j)$ and $\overline{\mathcal{Q}}(\mathbf{x}_j, \mathbf{p}_j, \mathbf{x}_k, \mathbf{p}_k)$ is the operator $\overline{\mathcal{Q}}$ acting on the variables $(\mathbf{x}_j, \mathbf{p}_j, \mathbf{x}_k, \mathbf{p}_k)$, namely

$$\begin{aligned}
&\overline{\mathcal{Q}}(\mathbf{x}_j, \mathbf{p}_j, \mathbf{x}_k, \mathbf{p}_k) F^{(n)}(\prod_{i=1}^{n}(\mathbf{x}_i, \mathbf{p}_i)) \\
&= \mathbb{E}\left\{\left[\prod_{i\neq j,k} W_z(\mathbf{x}_i, \mathbf{p}_i)\right] \int e^{i\mathbf{q}\cdot(\mathbf{x}_j-\mathbf{x}_k)} \Phi_{(\eta,\infty)}(0, \mathbf{q})\gamma^{-2}\right. \\
&\quad \times [W_z(\mathbf{x}_j, \mathbf{p}_j - \gamma\mathbf{q}/2) - W_z(\mathbf{x}_j, \mathbf{p}_j + \gamma\mathbf{q}/2)] \\
&\quad \left.\times\, [W_z(\mathbf{x}_k, \mathbf{p}_k - \gamma\mathbf{q}/2) - W_z(\mathbf{x}_k, \mathbf{p}_k + \gamma\mathbf{q}/2)]\; d\mathbf{q}\right\}.
\end{aligned}$$

Equation (45) can be more conveniently written as

$$\frac{\partial F^{(n)}}{\partial z} = \frac{1}{\tilde{k}} \sum_{j=1}^{n} \mathbf{p}_j \cdot \nabla_{\mathbf{x}_j} F^{(n)} + \tilde{k}^2 \sum_{j,k=1}^{n} \overline{\mathcal{Q}}(\mathbf{x}_j, \mathbf{p}_j, \mathbf{x}_k, \mathbf{p}_k) F^{(n)} \tag{46}$$

with the identification $\overline{\mathcal{Q}}(\mathbf{x}_j, \mathbf{p}_j, \mathbf{x}_j, \mathbf{p}_j) = \overline{\mathcal{Q}}_0(\mathbf{x}_j, \mathbf{p}_j)$. The operator

$$\mathcal{Q}_{\mathrm{sum}} = \sum_{j,k=1}^{n} \overline{\mathcal{Q}}(\mathbf{x}_j, \mathbf{p}_j, \mathbf{x}_k, \mathbf{p}_k) \tag{47}$$

is a non-positive symmetric operator. We note that the mean Wigner distribution can be exactly solved for from Eq. (46) for $n = 1$ [12] and has a number of interesting applications in optics including time reversal. The 2^{nd} moment equation $n = 2$ is related to the problem of scintillation [24] (see, e.g., [5]).

The uniqueness for Eq. (45) with any initial data

$$F^{(n)}(z = 0, \mathbf{x}_1, \mathbf{p}_1, \ldots, \mathbf{x}_n, \mathbf{p}_n) = \mathbb{E}\left[W_0(\mathbf{x}_1, \mathbf{p}_1) \cdots W_0(\mathbf{x}_n, \mathbf{p}_n)\right], \quad W_0 \in L^2(\mathbb{R}^{2d})$$

in the case of the Wigner-Moyal equation can be easily established by observing that the operator given by (47) is self-adjoint. For instance, for $n = 2$, we have that

$$\mathcal{F}_2^{-1}\overline{\mathcal{Q}}F^{(2)}(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) = \mathcal{F}_2^{-1}\overline{\mathcal{Q}}(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2)\mathcal{F}_2^{-1}F^{(2)}(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2),$$

where

$$\mathcal{F}_2^{-1}F^{(2)}(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) = \mathbb{E}\left[\mathcal{F}_2^{-1}W_z(\mathbf{x}_1, \mathbf{y}_1)\mathcal{F}_2^{-1}W_z(\mathbf{x}_2, \mathbf{y}_2)\right]$$

and $\mathcal{F}_2^{-1}\overline{\mathcal{Q}}(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2)$ is the *function*

$$\begin{aligned}
&\int e^{i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\Phi_{(\eta,\infty)}(0, \mathbf{q})\gamma^{-2}\left[e^{i\gamma\mathbf{y}_1\cdot\mathbf{q}/2} - e^{-i\gamma\mathbf{y}_1\cdot\mathbf{q}/2}\right]\left[e^{i\gamma\mathbf{y}_2\cdot\mathbf{q}/2} - e^{-i\gamma\mathbf{y}_2\cdot\mathbf{q}/2}\right] d\mathbf{q} \\
&= -8\gamma^{-2}\int \cos\left[\mathbf{q}\cdot(\mathbf{x}_1 - \mathbf{x}_2)\right]\Phi_{(\eta,\infty)}(0, \mathbf{q})\sin\left[\gamma\mathbf{y}_1\cdot\mathbf{q}/2\right]\sin\left[\gamma\mathbf{y}_2\cdot\mathbf{q}/2\right]d\mathbf{q}.
\end{aligned}$$

Namely, in the $(\mathbf{x}_j, \mathbf{y}_j)$ variables, the operator $\mathcal{Q}_{\mathrm{sum}}$ becomes the multiplication by a function which is dominated by the "diagonal terms" with $j = k$,

$$\mathcal{F}_2^{-1}\overline{\mathcal{Q}}_0F^{(2)}(\mathbf{x}_j, \mathbf{y}_j) = -8\gamma^{-2}\int \Phi_{(\eta,\infty)}(0, \mathbf{q})\sin^2\left[\gamma\mathbf{y}_j\cdot\mathbf{q}/2\right]d\mathbf{q}$$

and hence is non-positive. Therefore $\mathcal{Q}_{\mathrm{sum}}$ is a non-positive self-adjoint operator on L^2. The case with $n > 2$ is similar.

Each of the operators on the right side of (46) generates a unique C_0-semigroup of contractions on $L^2(\mathbb{R}^{2nd})$ and, by the product formula, their sum generates a unique C_0-semigroup of contractions on $L^2(\mathbb{R}^{2nd})$. Standard theory for linear equations then yields the uniqueness result for the weak solution of (46).

In the case of the Liouville equation, Eq. (46) can be more explicitly written as the Fokker-Planck equation on the phase space

$$\frac{\partial F^{(n)}}{\partial z} = \frac{1}{\tilde{k}} \sum_{j=1}^{n} \mathbf{p}_j \cdot \nabla_{\mathbf{x}_j} F^{(n)} + \tilde{k}^2 \sum_{j,k=1}^{n} \mathbf{D}(\mathbf{x}_j - \mathbf{x}_k) : \nabla_{\mathbf{p}_j} \nabla_{\mathbf{p}_k} F^{(n)} \tag{48}$$

with

$$\mathbf{D}(\mathbf{x}_j - \mathbf{x}_k) = \int e^{i\mathbf{q}\cdot(\mathbf{x}_j - \mathbf{x}_k)} \Phi_\eta^\rho(\mathbf{q}) \mathbf{q} \otimes \mathbf{q} d\mathbf{q}$$

with $\eta \geq 0$. In the worst case scenario allowed by the bound (7) (cf. (8)) the diffusion coefficient $D(0)$ diverges as $\rho \to \infty$ (but well-defined as $\eta \to 0$) when $H < 1/2$. When $H > 1/2$ then the limit $\rho \to 0$ poses no difficulty.

Moreover the diffusion operator

$$\sum_{j,k=1}^{n} \mathbf{D}(\mathbf{x}_j - \mathbf{x}_k) : \nabla_{\mathbf{p}_j} \nabla_{\mathbf{p}_k}$$

is an essentially self-adjoint positive operator on $C_c^\infty(\mathbb{R}^{2nd}) \subset L^2(\mathbb{R}^{2nd})$ due to the sub-Lipschitz growth of the square-root of $\mathbf{D}(\mathbf{x}_k - \mathbf{x}_k)$ at large $|\mathbf{x}_j|$, $|\mathbf{x}_k|$ [8]. The uniqueness follows from the same argument as in the previous case.

3. Assumptions and Main Theorems

3.1. Assumptions and properties of the refractive index field. As mentioned in the introduction, we assume that $V_z(\mathbf{x})$ is a square-integrable, z-stationary, $\mathbf{x}$-homogeneous process with a spectral density satisfying the upper bound (7).

Let $r(t)$ be a non-negative (random or deterministic) function such that

$$\begin{aligned} \left|\mathbb{E}\left[\mathbb{E}_z[V_s(\mathbf{x})]\mathbb{E}_z[V_t(\mathbf{y})]\right]\right| &= \left|\mathbb{E}\left[\mathbb{E}_z[V_s(\mathbf{x})]V_t(\mathbf{y})\right]\right| \\ &\leq r(s-z)r(t-z)\mathbb{E}\left[V_z^2\right], \quad \forall s,t \geq z, \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^d. \end{aligned} \tag{49}$$

An obvious candidate for $r(t)$ is the correlation coefficient defined as follows. Let $\mathcal{F}_z$ and $\mathcal{F}_z^+$ be the sigma-algebras generated by $\{V_s : \forall s \leq z\}$ and $\{V_s : \forall s \geq z\}$, respectively. The correlation coefficient $r_{\eta,\rho}(t)$ is given by

$$r_{\eta,\rho}(t) = \sup_{\substack{h\in\mathcal{F}_z \\ \mathbb{E}[h]=0,\mathbb{E}[h^2]=1}} \sup_{\substack{g\in\mathcal{F}_{z+t}^+ \\ \mathbb{E}[g]=0,\mathbb{E}[g^2]=1}} \mathbb{E}[hg]. \tag{50}$$

Lemma 1. *The correlation coefficient $r_{\eta,\rho}(t)$ as given by (50) satisfies the inequality (49).*

Proof. Let

$$h_s(\mathbf{x}) = \mathbb{E}_z[V_s(\mathbf{x})], \quad g_t(\mathbf{x}) = V_t(\mathbf{x}).$$

Clearly

$$h_s \in L^2(P, \Omega, \mathcal{F}_z),$$
$$g_t \in \in L^2(P, \Omega, \mathcal{F}_t^+),$$

and their second moments are uniformly bounded in $\mathbf{x}$ since

$$\mathbb{E}[h_s^2](\mathbf{x}) \leq \mathbb{E}[g_s^2](\mathbf{x}),$$
$$\mathbb{E}[g_s^2](\mathbf{x}) = \int \Phi(\xi, \mathbf{q}) d\xi d\mathbf{q}.$$

From the definition (50) we have

$$|\mathbb{E}[h_s(\mathbf{x})h_t(\mathbf{y})]| = |\mathbb{E}[h_s g_t]| \leq r_{\eta,\rho}(t-z)\mathbb{E}^{1/2}\left[h_s^2(\mathbf{x})\right]\mathbb{E}^{1/2}\left[g_t^2\right].$$

Hence by setting $s = t$ first and the Cauchy-Schwartz inequality we have

$$\mathbb{E}\left[h_s^2\Big(\mathbf{x})\right] \leq r_{\eta,\rho}^2(s-z)\mathbb{E}[g_t^2],$$
$$\mathbb{E}\left[h_s(\mathbf{x})h_t(\mathbf{y})\right] \leq r_{\eta,\rho}(t-z)r_{\eta,\rho}(s-z)\mathbb{E}[g_t^2], \quad \forall s, t \geq z, \forall \mathbf{x}, \mathbf{y}.$$

□

We assume

Assumption 1. *The function $r(t)$ in (49) satisfies*

$$\int_0^\infty \int_0^\infty r(s)r(t)dsdt < \infty.$$

Corollary 1. *The formula*

$$\tilde{V}_z(\mathbf{x}) = \int_z^\infty \mathbb{E}_z\left[V_s(\mathbf{x})\right]\, ds \tag{51}$$

defines a square-integrable z-stationary, $\mathbf{x}$-homogeneous process.

Proof. Let $\omega \in \Omega$ denote the random element and $\tau_{\vec{\mathbf{x}}}, \vec{\mathbf{x}} = (z, \mathbf{x}) \in \mathbb{R}^{d+1}$ the translation operator acting on Ω. Then without loss of generality we may assume that there exists a square-integrable function V defined on Ω such that

$$V_z(\mathbf{x}, \omega) = V(\tau_{\vec{\mathbf{x}}}\omega).$$

It suffices to show that the second moment of

$$\tilde{V}(\omega) \equiv \int_0^\infty \mathbb{E}_0\left[V(\tau_{(s,0)}\omega)\right]\, ds$$

is finite since

$$\tilde{V}_z(\mathbf{x}, \omega) = \tilde{V}(\tau_{\vec{\mathbf{x}}}\omega), \quad \forall \vec{\mathbf{x}} = (z, \mathbf{x}) \in \mathbb{R}^{d+1}.$$

To this end we have

$$\begin{aligned}\mathbb{E}\left[\tilde{V}^2\right] &= \mathbb{E}\left[\int_0^\infty \int_0^\infty \mathbb{E}_0[V_s(0)]\mathbb{E}_0[V_t(0)]dsdt\right]\\ &= \mathbb{E}\left[\int_0^\infty \int_0^\infty \mathbb{E}_0[V_s(0)]V_t(0)dsdt\right]\\ &\le \int_0^\infty \int_0^\infty r(s)r(t)dsdt\mathbb{E}[V_0^2]\end{aligned}$$

which is finite by Assumption 1. □

One can adopt other alternative mixing coefficients to get the above results and higher order moment estimates, see Appendix A. Hereafter we will mainly focus on the correlation coefficient as it is most convenient to work within the Gaussian case and we shall write explicitly the dependence of the correlation coefficient on η, ρ as $r_{\eta,\rho}(t)$.

In the Gaussian case the correlation coefficient $r_{\eta,\rho}(t)$ equals the linear correlation coefficient given by

$$r_{\eta,\rho}(t) = \sup_{g_1,g_2} \int R(t-\tau_1-\tau_2,\mathbf{k})g_1(\tau_1,\mathbf{k})g_2(\tau_2,\mathbf{k})d\mathbf{k}d\tau_1 d\tau_2, \tag{52}$$

where

$$R(t,\mathbf{k}) = \int e^{it\xi}\Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi,$$

and the supremum is taken over all g_1, $g_2 \in L^2(\mathbb{R}^{d+1})$ which are supported on $(-\infty,0]\times\mathbb{R}^d$ and satisfy the constraint

$$\begin{aligned}&\int R(t-t',\mathbf{k})g_1(t,\mathbf{k})\bar{g}_1(t',\mathbf{k})dtdt'd\mathbf{k}\\ &\quad= \int R(t-t',\mathbf{k})g_2(t,\mathbf{k})\bar{g}_2(t',\mathbf{k})dtdt'd\mathbf{k} = 1.\end{aligned} \tag{53}$$

Alternatively, by the Paley-Wiener theorem we can write

$$r_{\eta,\rho}(t) = \sup_{f_1,f_2} \int e^{i\xi t} f_1(\xi,\mathbf{k}) f_2(\xi,\mathbf{k})\Phi_{\eta,\rho}(\xi,\mathbf{k})d\xi d\mathbf{k}, \tag{54}$$

where f_1, f_2 are elements of the Hardy space $\mathcal{H}^2$ of $L^2(\Phi_{(\eta,\rho)}d\xi d\mathbf{k})$-valued analytic functions in the upper half ξ-space satisfying the normalization condition

$$\int |f_j(\xi,\mathbf{k})|^2\Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k} = 1, \quad j=1,2.$$

There are various criteria for the decay rate of the linear correlation coefficients, see [17].

Corollary 2. *If V_z is a Gaussian random field and its linear correlation coefficient $r_{\eta,\rho}(t)$ is integrable, then $\tilde{V}_z$ is also Gaussian and hence possesses finite moments of all orders.*

This follows from the fact that the mapping from V_z to $\tilde{V}_z$ is a bounded linear operator on the Gaussian space.

The main property of $\tilde{V}_z$ as a random function is that

$$\mathcal{A}\tilde{V}_z = -V_z, \quad \text{a.s.} \quad z \in \mathbb{R}. \tag{55}$$

Since $\mathcal{A}$ commutes with the shift in $\mathbf{x}$ so the appearance of $\mathbf{x}$ in Eq. (55) is suppressed.

We have the following simple relation

$$\begin{aligned}\lim_{\lambda\to\infty} \mathbb{E}\left[\tilde{V}_{z\lambda}(\mathbf{x})V_{z\lambda}(\mathbf{y})\right] &= \lim_{\lambda\to\infty} \int e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{p}} \int \frac{1}{i\xi}\left(e^{iz\lambda\xi}-1\right)\Phi_{(\eta,\rho)}(\xi,\mathbf{p})d\xi d\mathbf{p} \\ &= \pi \int e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{p}}\Phi_{(\eta,\rho)}(0,\mathbf{p})d\mathbf{p}, \quad \forall z. \end{aligned} \tag{56}$$

Define the covariance functions

$$\tilde{B}_z(\mathbf{x}-\mathbf{y}) \equiv \mathbb{E}\left[\tilde{V}_z(\mathbf{x})\tilde{V}_z(\mathbf{y})\right]$$

and write

$$\tilde{B}_z(\mathbf{x}) = \int e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{\Phi}_z(\mathbf{k})d\mathbf{k},$$

where $\tilde{\Phi}_z(\mathbf{k})$ is its spectral density function.

By the properties of the orthogonal projection $\mathbb{E}_z[\cdot]$, we know that

$$\mathbb{E}\left[\mathbb{E}_z[\hat{V}(A)]\mathbb{E}_z[\hat{V}(A)]\right] \le \mathbb{E}\left[\hat{V}(A)\hat{V}(A)\right] = \int_A \Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k} \tag{57}$$

for every Borel set $A \subset \mathbb{R}^{d+1}$.

Assumption 2. *For any* $\eta > 0$*,*

$$R_\eta = \limsup_{\rho\to\infty} \int_0^\infty r_{\eta,\rho}(t)dt < \infty$$

such that

$$\limsup_{\eta\to 0} \eta R_\eta < \infty.$$

For Gaussian fields with the generalized von Kármán spectrum (8), a straightforward scaling argument with (54) shows that

$$r_{\eta,\infty}(t) = r_{1,\infty}(\eta t),$$

hence

$$R_\eta = \eta^{-1}R_1.$$

This motivates Assumption 2.

Set

$$\tilde{\Phi}^\varepsilon_z(\mathbf{k}) \equiv \tilde{\Phi}_{\varepsilon^{-2}z}(\xi,\mathbf{k})$$

which is the spectral density of $\tilde{V}^\varepsilon_z(\mathbf{x}) \equiv \tilde{V}_{z/\varepsilon^2}(\mathbf{x})$.

Define analogously to (22)

$$\tilde{\mathcal{L}}_z^\varepsilon \theta(\mathbf{x}, \mathbf{p}) \equiv -i\gamma^{-1}\mathcal{F}_2 \left[\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x}, \mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x}, \mathbf{y})\right] \tag{58}$$

with

$$\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x}, \mathbf{y}) \equiv \tilde{V}_z^\varepsilon(\mathbf{x} + \gamma\mathbf{y}/2) - \tilde{V}_z^\varepsilon(\mathbf{x} - \gamma\mathbf{y}/2).$$

Lemma 2 (Appendix B)**.** *For each $z_0 < \infty$ there exists a positive constant $\tilde{C} < \infty$ such that*

$$\sup_{\substack{|z|\le z_0\\ |\mathbf{y}|\le L}} \mathbb{E}\left[\left(\delta_\gamma V_z^\varepsilon\right)^2\right](\mathbf{y}) \le \tilde{C}\gamma^2 \left|\min(\gamma^{-1}, \rho)\right|^{2-2H},$$

$$\sup_{|z|\le z_0} \mathbb{E}\left[\tilde{V}_z^\varepsilon(\mathbf{x})\right]^2 \le \tilde{C}\eta^{-2-2H},$$

$$\sup_{\substack{|z|\le z_0\\ |\mathbf{y}|\le L}} \mathbb{E}\left[\left(\delta_\gamma \tilde{V}_z^\varepsilon\right)^2\right](\mathbf{y}) \le \tilde{C}\eta^{-2}\gamma^2|\min(\rho, \gamma^{-1})|^{2-2H},$$

$$\sup_{\substack{|z|\le z_0\\ |\mathbf{y}|\le L}} \left|\nabla_{\mathbf{y}}\mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right| \le \tilde{C}\eta^{-2}\gamma^2\rho^{1-H}|\min(\rho, \gamma^{-1})|^{1-H},$$

$$\sup_{|z|\le z_0} \mathbb{E}\|\mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^\varepsilon\theta)\|_2^2 \le \tilde{C}\eta^{-2}\rho^{4-2H}, \quad \theta \in \mathcal{S}$$

for all $H \in (0, 1)$, $\varepsilon, \gamma, \eta \le 1 \le \rho$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, where the constant $\tilde{C}$ depends only on z_0, L and θ.

We also need to know the first few moments the random fields involved. The case of Gaussian fields motivates the following assumption of the 6^{th} order sub-Gaussian property.

Assumption 3.

$$\sup_{|\mathbf{y}|\le L} \mathbb{E}\left[\delta_\gamma V_z^\varepsilon(\mathbf{y})\right]^4 \le C_1 \sup_{|\mathbf{y}|\le L} \mathbb{E}^2\left[\delta_\gamma V_z^\varepsilon\right]^2(\mathbf{y}), \tag{59}$$

$$\sup_{|\mathbf{y}|\le L} \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^4(\mathbf{y}) \le C_2 \sup_{|\mathbf{y}|\le L} \mathbb{E}^2\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2(\mathbf{y}), \tag{60}$$

$$\sup_{|\mathbf{y}|\le L} \mathbb{E}\left[\left[\delta_\gamma V_z^\varepsilon\right]^2\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^4\right](\mathbf{y}) \le C_3 \left(\sup_{|\mathbf{y}|\le L} \mathbb{E}\left[\delta_\gamma V_z^\varepsilon\right]^2(\mathbf{y})\right) \times \left(\sup_{|\mathbf{y}|\le L} \mathbb{E}^2\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right) \tag{61}$$

for all $L < \infty$, where the constants C_1, C_2 and C_3 are independent of $\varepsilon, \eta, \rho, \gamma$.

From (22) and (58) we can form the iteration of operators $\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon$,

$$\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x}, \mathbf{p}) = -\gamma^{-2}\mathcal{F}_2\left[\delta_\gamma V_z^\varepsilon(\mathbf{x}, \mathbf{y})\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x}, \mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x}, \mathbf{y})\right].$$

The operator $\mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta$ is well-defined if $\delta_\gamma V_z^\varepsilon$ and $\delta_\gamma \tilde{V}_z^\varepsilon$ are locally square-integrable. Other iterations of $\mathcal{L}_z^\varepsilon$ and $\tilde{\mathcal{L}}_z^\varepsilon$ allowed by Assumption 3 can be similarly constructed.

The following estimates can be obtained from Lemma 2 and Assumption 3.

Corollary 3 (Appendix C)**.**

$$\mathbb{E}\left[\|\mathcal{L}_z^\varepsilon \theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^\varepsilon \theta(\mathbf{y},\mathbf{q})\|_2^2\right] \le C\left(\eta^{-2}|\min(\rho,\gamma^{-1})|^{4-4H}\right),$$
$$\mathbb{E}\left[\|\mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\|_2^2\right] \le C\left(\eta^{-2}|\min(\rho,\gamma^{-1})|^{4-4H}\right),$$
$$\mathbb{E}\left[\|\tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\|_2^2\right] \le C\left(\eta^{-4}|\min(\rho,\gamma^{-1})|^{4-4H}\right),$$
$$\mathbb{E}\left\|\mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\right\|_2^2 \le C\left(\eta^{-4}|\min(\rho,\gamma^{-1})|^{6-6H}\right),$$

where the constant C is independent of ρ, η, γ and L is the radius of the ball containing the support of $\mathcal{F}_2^{-1}\theta$.

Assumption 4. *For every $\theta \in \mathcal{S}$, there exists a random constant C_5 such that*

$$\sup_{z<z_0} \|\delta_\gamma \tilde{V}_z^\varepsilon \mathcal{F}_2^{-1}\theta\|_4 \le \frac{C_5}{\sqrt{\varepsilon}} \sup_{\substack{z\in[0,z_0]\\ |\mathbf{x}|,|\mathbf{y}|\le L}} \mathbb{E}^{1/2}|\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})|^2, \quad \forall\theta\in\mathcal{S}, \varepsilon,\eta,\gamma \le 1 \le \rho \tag{62}$$

with C_5 possessing finite moments and depending only on θ, z_0, where L is the radius of the ball containing the support of $\mathcal{F}_2^{-1}\theta$, cf. Lemma 2 and (63).

For a Gaussian random field, Assumption 4 is readily satisfied by Lemma 2 and Borell's inequality [1]

$$\sup_{z<z_0} \|\delta_\gamma \tilde{V}_z^\varepsilon \mathcal{F}_2^{-1}\theta\|_4 \le \|\mathcal{F}_2^{-1}\theta\|_4 \sup_{\substack{z\in[0,z_0]\\ |\mathbf{x}|,|\mathbf{y}|\le L}} |\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})|$$
$$\le C_5 \log\left(\frac{z_0}{\varepsilon^2}\right) \sup_{\substack{z\in[0,z_0]\\ |\mathbf{x}|,|\mathbf{y}|\le L}} \mathbb{E}^{1/2}|\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})|^2, \quad \forall\eta,\gamma\le 1\le\rho, \tag{63}$$

where the random constant C_5 has a Gaussian-like tail.

Note that with γ or ρ held fixed the first term on the right side of (62) is always $O(1)$. Compared to the corresponding condition (63) for the Gaussian field condition (62) allows for a certain degree of intermittency in the refractive index field.

As we have seen above, most of the assumptions here are motivated by the Gaussian case and we have formulated them in such a way as to allow a significant level of non-Gaussian fluctuation.

3.2. Main theorems.

Theorem 1. *Let V_z^ε be a z-stationary,* **x***-homogeneous, almost surely locally bounded random process with the spectral density satisfying the bound (7) and Assumptions 1, 2, 3, 4. Let $\gamma > 0$ be fixed.*

(i) *Let η be fixed and ρ be fixed or tend to ∞ as $\varepsilon \to 0$ such that*

$$\lim_{\varepsilon\to 0} \varepsilon\rho^{2-H} = 0. \tag{64}$$

Then the weak solution W^ε of the Wigner-Moyal equation with the initial condition $W_0 \in L^2(\mathbb{R}^{2d})$ converges in law in the space $D([0,\infty); L^2_w(\mathbb{R}^{2d}))$ of L^2-valued right continuous processes with left limits endowed with the Skorohod topology to that of the corresponding Gaussian white-noise model with the covariance operators $\overline{Q}$ and $\overline{Q}_0$ as given by (33) and (32), respectively (see also (42) and (43)). The statement holds true for any $H \in (0,1)$.

(ii) *Suppose additionally that $H < 1/2$ and $\eta = \eta(\varepsilon) \to 0$ such that*

$$\lim_{\varepsilon\to 0} \varepsilon\eta^{-1}(\eta^{-1} + \rho^{2-H}) = 0. \tag{65}$$

Then the same convergence holds true.

Here and below $L^2_w(\mathbb{R}^{2d})$ is the space of square integrable functions on the phase space $\mathbb{R}^{2d}$ endowed with the weak topology.

The next theorem concerns a similar convergence to the solution of a Gaussian white-noise model for the Liouville equation.

Theorem 2. *Let V^ε_z be a z-stationary, **x**-homogeneous, almost surely smooth, locally bounded random process with the spectral density satisfying the bound (7) and Assumptions 1, 2, 3, 4.*

Let $\gamma = \gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then under any of the following three sets of conditions:
(i) *$\rho < \infty$ and $\eta > 0$ held fixed;*
(ii) *$H > 1/2$, $\eta > 0$ fixed and $\rho = \rho(\varepsilon) \to \infty$ as $\varepsilon \to 0$ such that*

$$\lim_{\varepsilon\to 0} \varepsilon\rho^{2-H} = 0; \tag{66}$$

(iii) *$H < 1/2$, $\rho < \infty$ fixed and $\eta = \eta(\varepsilon) \to 0$ such that*

$$\lim_{\varepsilon\to 0} \varepsilon\eta^{-2} = 0; \tag{67}$$

the weak solutions W^ε of the Wigner-Moyal equation (16) with the initial condition $W_0 \in L^2(\mathbb{R}^{2d})$ converge in distribution in the space $D([0,\infty); L^2_w(\mathbb{R}^{2d}))$ to the martingale solution of the Liouville equation of the Gaussian white-noise model with the covariance operators $\overline{Q}$ and $\overline{Q}_0$ as given by (34) and (35), respectively (see also (42) and (44)).

It is worthwhile to point out that the stochastic geometrical optics limit (Theorem 2) puts restriction more on the aspect ratio ε^2 of the wave beam than on the ratio $\sqrt{r}\rho$ between the Fresnel length and the inner scale as commonly assumed in the literature (see for example [22]).

Note also that the Kolmogorov value $H = 1/3$ is covered by the regimes of Theorem 1 and Theorem 2(i), (iii).

Remark 2. Both Theorem 1 and 2 can be viewed as a construction (and the convergence) of approximate solutions (via Remark 1) to the Gaussian white-noise models which are widely used in practical applications [24, 5].

4. Proof of Theorem 1 and 2

4.1. Tightness. In the sequel we will adopt the following notation:

$$f_z \equiv f(\langle W_z^\varepsilon, \theta \rangle), \quad f'_z \equiv f'(\langle W_z^\varepsilon, \theta \rangle), \quad f''_z \equiv f''(\langle W_z^\varepsilon, \theta \rangle), \qquad \forall f \in C^\infty(\mathbb{R}). \quad (68)$$

Namely, the prime stands for the differentiation w.r.t. the original argument (not z) of f, f', etc. Let L denote the radius of the ball containing the support of $\mathcal{F}_2^{-1}\theta$. Let all the constants $c, c', c_1, c_2, \dots$ etc. in the sequel be independent of ρ, η, γ and ε and depend only on $z_0, \theta, \|W_0\|_2$ and f.

First we note that since $\mathcal{S}$ is dense in $L^2(\mathbb{R}^{2d})$ and $\|W_z^\varepsilon\|_2 \le \|W_0\|_2, \forall z > 0$, the tightness of the family of $L^2(\mathbb{R}^{2d})$-valued processes $\{W^\varepsilon, 0 < \varepsilon < 1\}$ in $D([0,\infty); L_w^2(\mathbb{R}^{2d})$ is equivalent to the tightness of the family in $D([0,\infty); \mathcal{S}')$ as distribution-valued processes. According to [14], a family of processes $\{W^\varepsilon, 0 < \varepsilon < 1\} \subset D([0,\infty); \mathcal{S}')$ is tight if and only if for every test function $\theta \in \mathcal{S}$ the family of processes $\{\langle W^\varepsilon, \theta\rangle, 0 < \varepsilon < 1\} \subset D([0,\infty); \mathbb{R})$ is tight. With this remark we can now use the tightness criterion of [19] (Chap. 3, Theorem 4) for finite dimensional processes, namely, we will prove: Firstly,

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}\{\sup_{z<z_0} |\langle W_z^\varepsilon, \theta\rangle| \ge N\} = 0, \quad \forall z_0 < \infty. \quad (69)$$

Secondly, for each $f \in C^\infty(\mathbb{R})$ there is a sequence $f_z^\varepsilon \in \mathcal{D}(\mathcal{A}^\varepsilon)$ such that for each $z_0 < \infty\{\mathcal{A}^\varepsilon f_z^\varepsilon, 0 < \varepsilon < 1, 0 < z < z_0\}$ is uniformly integrable and

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\sup_{z<z_0} |f_z^\varepsilon - f(\langle W_z^\varepsilon, \theta\rangle)| \ge \delta\} = 0, \quad \forall \delta > 0. \quad (70)$$

Then it follows that the laws of $\{\langle W^\varepsilon, \theta\rangle, 0 < \varepsilon < 1\}$ are tight in the space of $D([0,\infty); \mathbb{R})$ and hence $\{W_z^\varepsilon\}$ is tight in $D([0,\infty); L_w^2(\mathbb{R}^{2d}))$.

Condition (69) is satisfied because the L^2-norm is preserved.

We shall construct a test function of the form $f_z^\varepsilon = f_z + f_{1,z}^\varepsilon + f_{2,z}^\varepsilon + f_{3,z}^\varepsilon$. First we construct the first perturbation $f_{1,z}^\varepsilon$. Let

$$\tilde{V}_z^\varepsilon = \tilde{V}_{z/\varepsilon^2}.$$

Recall that

$$\mathcal{A}^\varepsilon \tilde{V}_z^\varepsilon = -\varepsilon^{-2} V_z^\varepsilon.$$

Let

$$\begin{aligned} f_{1,z}^\varepsilon &\equiv \frac{\tilde{k}}{\varepsilon} \int_z^\infty f'_z \langle W_z^\varepsilon, \mathbb{E}_z^\varepsilon \mathcal{L}_s^\varepsilon \theta\rangle \, ds \\ &= \tilde{k}\varepsilon f'_z \left\langle \mathcal{F}_2^{-1} W_z^\varepsilon, \gamma^{-1}\delta_\gamma \int_z^\infty \mathbb{E}_z[V_s^\varepsilon] ds \mathcal{F}_2^{-1}\theta \right\rangle \\ &= \tilde{k}\varepsilon f'_z \left\langle \mathcal{F}_2^{-1} W_z^\varepsilon, \gamma^{-1}\delta_\gamma \tilde{V}_z^\varepsilon \mathcal{F}_2^{-1}\theta \right\rangle \\ &= \tilde{k}\varepsilon f'_z \left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon \theta \right\rangle \end{aligned} \quad (71)$$

be the 1st perturbation of f_z.

Proposition 1.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f^\varepsilon_{1,z}| = 0, \quad \lim_{\varepsilon\to 0}\sup_{z<z_0}|f^\varepsilon_{1,z}| = 0 \quad \textit{in probability}\,.$$

Proof. First

$$\mathbb{E}[|f^\varepsilon_{1,z}|] \le \varepsilon\|f'\|_\infty\|W_0\|_2\mathbb{E}\|\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2 \tag{72}$$

$$\le c\varepsilon\|f'\|_\infty\|W_0\|_2 \sup_{|\mathbf{x}|,|\mathbf{y}|\le L}\mathbb{E}^{1/2}\left[\gamma^{-1}\delta_\gamma\tilde{V}^\varepsilon_z(\mathbf{x},\mathbf{y})\right]^2$$

$$= O\left(\varepsilon\eta^{-1}|\min(\rho,\gamma^{-1})|^{1-H}\right) \tag{73}$$

which is of the following order of magnitude:

$$\begin{cases} \varepsilon, & \text{if } \eta, \rho \text{ held fixed} \\ \varepsilon, & \text{if } \gamma, \eta \text{ held fixed} \\ \varepsilon\eta^{-1}, & \text{if } \gamma \text{ or } \rho \text{ held fixed} \\ \varepsilon|\min(\rho,\gamma^{-1})|^{1-H}, & \text{if } \eta \text{ is held fixed,} \end{cases} \tag{74}$$

and vanishes in the respective regimes. Secondly, we have

$$\sup_{z<z_0}|f^\varepsilon_{1,z}| \le \varepsilon\|f'\|_\infty\|W_0\|_2\sup_{z<z_0}\gamma^{-1}\|\delta_\gamma\tilde{V}^\varepsilon_z\mathcal{F}_2^{-1}\theta\|_2$$

$$\le c\varepsilon^{1/2}\sup_{|\mathbf{x}|,|\mathbf{y}|\le L}\mathbb{E}^{1/2}|\gamma^{-1}\delta_\gamma\tilde{V}^\varepsilon_z(\mathbf{x},\mathbf{y})|^2$$

$$= c'\varepsilon^{1/2}\eta^{-1}|\min(\rho,\gamma^{-1})|^{1-H} \tag{75}$$

by Assumption 4, with a random constant c' possessing finite moments. The right side of (75) is of the following order of magnitude:

$$\begin{cases} \varepsilon^{1/2}, & \text{if } \eta, \rho \text{ held fixed} \\ \varepsilon^{1/2}, & \text{if } \gamma, \eta \text{ held fixed} \\ \varepsilon^{1/2}\eta^{-1}, & \text{if } \rho \text{ or } \gamma \text{ held fixed} \\ \varepsilon^{1/2}|\min(\rho,\gamma^{-1})|^{1-H}, & \text{if } \eta \text{ is held fixed,} \end{cases} \tag{76}$$

which vanishes in the respective regimes. The right side of (75) now converges to zero in probability by a simple application of Chebyshev's inequality and (65). □

A straightforward calculation yields

$$\mathcal{A}^\varepsilon f^\varepsilon_1 = -\tilde{k}\varepsilon f'_z\left\langle W^\varepsilon_z, \left[\frac{\mathbf{p}}{\tilde{k}}\cdot\nabla + \frac{\tilde{k}}{\varepsilon}\mathcal{L}^\varepsilon_z\right]\tilde{\mathcal{L}}^\varepsilon_z\theta\right\rangle$$

$$-\frac{\tilde{k}}{\varepsilon}f'_z\left\langle W^\varepsilon_z, \mathcal{L}^\varepsilon_z\theta\right\rangle + \tilde{k}\varepsilon f''_z\left\langle W^\varepsilon_z, \mathcal{A}^\varepsilon\theta\right\rangle\left\langle W^\varepsilon_z, \tilde{\mathcal{L}}^\varepsilon_z\theta\right\rangle,$$

where $\mathcal{A}^\varepsilon\theta$ denotes

$$\mathcal{A}^\varepsilon\theta = -\frac{1}{\tilde{k}}\mathbf{p}\cdot\nabla_\mathbf{x}\theta - \frac{\tilde{k}}{\varepsilon}\mathcal{L}^\varepsilon_z\theta$$

cf. (28). Hence

$$\mathcal{A}^\varepsilon\left[f_z+f^\varepsilon_{1,z}\right]=\frac{1}{\tilde{k}}f'_z\left\langle W^\varepsilon_z,\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle+\tilde{k}^2f'_z\left\langle W^\varepsilon_z,\mathcal{L}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\right\rangle+\tilde{k}^2f''_z\left\langle W^\varepsilon_z,\mathcal{L}^\varepsilon_z\theta\right\rangle\left\langle W^\varepsilon_z,\tilde{\mathcal{L}}^\varepsilon_z\theta\right\rangle$$
$$+\varepsilon\left[f'_z\left\langle W^\varepsilon_z,\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}^\varepsilon_z\theta\right\rangle+f''_z\left\langle W^\varepsilon_z,\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle\left\langle W^\varepsilon_z,\tilde{\mathcal{L}}^\varepsilon_z\theta\right\rangle\right]$$
$$=A^\varepsilon_1(z)+A^\varepsilon_2(z)+A^\varepsilon_3(z)+R^\varepsilon_1(z),$$

where $A^\varepsilon_2(z)$ and $A^\varepsilon_3(z)$ are the coupling terms.

Proposition 2.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R^\varepsilon_1(z)|=0.$$

Proof. By Lemma 2 we have

$$|R^\varepsilon_1|\leq\varepsilon\|f''\|_\infty\|W_0\|_2^2\left[\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\|_2\|\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2+\|\mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}^\varepsilon_z\theta)\|_2\right]$$
$$=O\left(\eta^{-1}(|\min(\rho,\gamma^{-1})|^{1-H}+\rho^{2-H})\right),\tag{77}$$

which is of the following order of magnitude:

$$\begin{cases}\varepsilon, & \text{if } \eta,\rho \text{ held fixed}\\ \varepsilon\rho^{2-H}, & \text{if } \eta,\gamma \text{ held fixed}\\ \varepsilon\eta^{-1}, & \text{if } \rho \text{ is held fixed}\\ \varepsilon\eta^{-1}\rho^{2-H}, & \text{if } \gamma \text{ held fixed}\\ \varepsilon(|\min(\rho,\gamma^{-1})|^{1-H}+\rho^{2-H}), & \text{if } \eta \text{ held fixed}\end{cases}\tag{78}$$

and vanishes in the respective regimes. □

We introduce the next perturbations $f^\varepsilon_{2,z}$, $f^\varepsilon_{3,z}$. Let

$$A_2^{(1)}(\phi)\equiv\int\phi(\mathbf{x},\mathbf{p})\mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\phi(\mathbf{y},\mathbf{q})\,d\mathbf{x}d\mathbf{p}\,d\mathbf{y}d\mathbf{q},\tag{79}$$
$$A_1^{(1)}(\phi)\equiv\int\mathcal{Q}'_1\theta(\mathbf{x},\mathbf{p})\phi(\mathbf{x},\mathbf{p})\;d\mathbf{x}d\mathbf{p},\tag{80}$$

where

$$\mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})=\mathbb{E}\left[\mathcal{L}^\varepsilon_z\theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}^\varepsilon_z\theta(\mathbf{y},\mathbf{q})\right],\tag{81}$$

and

$$\mathcal{Q}'_1\theta(\mathbf{x},\mathbf{p})=\mathbb{E}\left[\mathcal{L}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta(\mathbf{x},\mathbf{p})\right],$$

where the operator $\tilde{\mathcal{L}}^\varepsilon_z$ is defined as in (58). Note that $\mathcal{Q}_1\theta$ and $\mathcal{Q}'_1\theta$ are $O(1)$ terms because of (56).

Clearly, we have

$$A_2^{(1)}(\phi)=\mathbb{E}\left[\left\langle\phi,\mathcal{L}^\varepsilon_z\theta\right\rangle\left\langle\phi,\tilde{\mathcal{L}}^\varepsilon_z\theta\right\rangle\right].\tag{82}$$

Define

$$f_{2,z}^{\varepsilon} \equiv \tilde{k}^2 f_z'' \int_z^{\infty} \mathbb{E}_z^{\varepsilon} \left[\langle W_z^{\varepsilon}, \mathcal{L}_s^{\varepsilon}\theta \rangle \left\langle W_z^{\varepsilon}, \tilde{\mathcal{L}}_s^{\varepsilon}\theta \right\rangle - A_2^{(1)}(W_z^{\varepsilon}) \right] ds,$$

$$f_{3,z}^{\varepsilon} \equiv \tilde{k}^2 f_z' \int_z^{\infty} \mathbb{E}_z^{\varepsilon} \left[\left\langle W_z^{\varepsilon}, \mathcal{L}_s^{\varepsilon}\tilde{\mathcal{L}}_s^{\varepsilon}\theta \right\rangle - A_3^{(1)}(W_z^{\varepsilon}) \right] ds.$$

Let

$$\mathcal{Q}_2(\theta \otimes \theta)(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \equiv \mathbb{E}\left[\tilde{\mathcal{L}}_z^{\varepsilon}\theta(\mathbf{x}, \mathbf{p}) \tilde{\mathcal{L}}_z^{\varepsilon}\theta(\mathbf{y}, \mathbf{q}) \right]$$

and

$$\mathcal{Q}_2'\theta(\mathbf{x}, \mathbf{p}) = \mathbb{E}\left[\tilde{\mathcal{L}}_z^{\varepsilon}\tilde{\mathcal{L}}_z^{\varepsilon}\theta(\mathbf{x}, \mathbf{p}) \right].$$

Let

$$A_2^{(2)}(\phi) \equiv \int \phi(\mathbf{x}, \mathbf{p}) \mathcal{Q}_2(\theta \otimes \theta)(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \phi(\mathbf{y}, \mathbf{q}) \; d\mathbf{x} d\mathbf{p}\, d\mathbf{y} d\mathbf{q}, \tag{83}$$

$$A_1^{(2)}(\phi) \equiv \int \mathcal{Q}_2'\theta(\mathbf{x}, \mathbf{p}) \phi(\mathbf{x}, \mathbf{p}) \; d\mathbf{x}\, d\mathbf{p}, \tag{84}$$

we then have

$$f_{2,z}^{\varepsilon} = \frac{\varepsilon^2 \tilde{k}^2}{2} f_z'' \left[\left\langle W_z^{\varepsilon}, \tilde{\mathcal{L}}_z^{\varepsilon}\theta \right\rangle^2 - A_2^{(2)}(W_z^{\varepsilon}) \right], \tag{85}$$

$$f_{3,z}^{\varepsilon} = \frac{\varepsilon^2 \tilde{k}^2}{2} f_z' \left[\left\langle W_z^{\varepsilon}, \tilde{\mathcal{L}}_z^{\varepsilon}\tilde{\mathcal{L}}_z^{\varepsilon}\theta \right\rangle - A_3^{(2)}(W_z^{\varepsilon}) \right]. \tag{86}$$

Proposition 3.

$$\lim_{\varepsilon \to 0} \sup_{z<z_0} \mathbb{E}|f_{j,z}^{\varepsilon}| = 0, \quad \lim_{\varepsilon \to 0} \sup_{z<z_0} |f_{j,z}^{\varepsilon}| = 0, \quad j = 2, 3.$$

Proof. We have the bounds

$$\sup_{z<z_0} \mathbb{E}|f_{2,z}^{\varepsilon}| \leq \sup_{z<z_0} \varepsilon^2 \tilde{k}^2 \|f''\|_{\infty} \left[\|W_0\|_2^2 \mathbb{E}\|\tilde{\mathcal{L}}_z^{\varepsilon}\theta\|_2^2 + \mathbb{E}[A_2^{(2)}(W_z^{\varepsilon})] \right],$$

$$\sup_{z<z_0} \mathbb{E}|f_{3,z}^{\varepsilon}| \leq \sup_{z<z_0} \varepsilon^2 \tilde{k}^2 \|f'\|_{\infty} \left[\|W_0\|_2 \mathbb{E}\|\tilde{\mathcal{L}}_z^{\varepsilon}\tilde{\mathcal{L}}_z^{\varepsilon}\theta\|_2 + \mathbb{E}[A_1^{(2)}(W_z^{\varepsilon})] \right].$$

The first term can be estimated as in (74); the second term can be estimated as in (74) by using (62).

As for estimating $\sup_{z<z_0} |f_{j,z}^{\varepsilon}|$, $j = 2, 3$, we have

$$\sup_{z<z_0} |f_{2,z}^{\varepsilon}| \leq \sup_{z<z_0} \varepsilon^2 \tilde{k}^2 \|f''\|_{\infty} \left[\|W_0\|_2^2 \|\tilde{\mathcal{L}}_z^{\varepsilon}\theta\|_2^2 + A_2^{(2)}(W_z^{\varepsilon}) \right],$$

$$\sup_{z<z_0} |f_{3,z}^{\varepsilon}| \leq \sup_{z<z_0} \varepsilon^2 \tilde{k}^2 \|f'\|_{\infty} \left[\|W_0\|_2 \|\tilde{\mathcal{L}}_z^{\varepsilon}\tilde{\mathcal{L}}_z^{\varepsilon}\theta\|_2 + A_1^{(2)}(W_z^{\varepsilon}) \right].$$

Using the assumption (62) we can estimate the right side of the above as in (76). □

We have

$$\mathcal{A}^\varepsilon f^\varepsilon_{2,z} = \tilde{k}^2 f''_z \left[-\left\langle W^\varepsilon_z, \mathcal{L}^\varepsilon_z \theta \right\rangle \left\langle W^\varepsilon_z, \tilde{\mathcal{L}}^\varepsilon_z \theta \right\rangle + A^{(1)}_2(W^\varepsilon_z) \right] + R^\varepsilon_2(z),$$
$$\mathcal{A}^\varepsilon f^\varepsilon_{3,z} = \tilde{k}^2 f'_z \left[-\left\langle W^\varepsilon_z, \mathcal{L}^\varepsilon_z \tilde{\mathcal{L}}^\varepsilon_z \theta \right\rangle + A^{(1)}_3(W^\varepsilon_z) \right] + R^\varepsilon_3(z),$$

with

$$\begin{aligned} R^\varepsilon_2(z) = {} & \varepsilon^2 \frac{\tilde{k}^2}{2} f'''_z \left[\frac{1}{\tilde{k}} \left\langle W^\varepsilon_z, \mathbf{p} \cdot \nabla_{\mathbf{x}} \theta \right\rangle + \frac{\tilde{k}}{\varepsilon} \left\langle W^\varepsilon_z, \mathcal{L}^\varepsilon_z \theta \right\rangle \right] \left[\left\langle W^\varepsilon_z, \tilde{\mathcal{L}}^\varepsilon_z \theta \right\rangle^2 - A^{(2)}_2(W^\varepsilon_z) \right] \\ & + \varepsilon^2 \tilde{k}^2 f''_z \left\langle W^\varepsilon_z, \tilde{\mathcal{L}}^\varepsilon_z \theta \right\rangle \left[\frac{1}{\tilde{k}} \left\langle W^\varepsilon_z, \mathbf{p} \cdot \nabla_{\mathbf{x}} (\tilde{\mathcal{L}}^\varepsilon_z \theta) \right\rangle + \frac{\tilde{k}}{\varepsilon} \left\langle W^\varepsilon_z, \mathcal{L}^\varepsilon_z \tilde{\mathcal{L}}^\varepsilon_z \theta \right\rangle \right] \\ & - \varepsilon^2 \tilde{k}^2 f'_z \left[\frac{1}{\tilde{k}} \left\langle W^\varepsilon_z, \mathbf{p} \cdot \nabla_{\mathbf{x}} (G^{(2)}_\theta W^\varepsilon_z) \right\rangle + \frac{\tilde{k}}{\varepsilon} \left\langle W^\varepsilon_z, \mathcal{L}^\varepsilon_z G^{(2)}_\theta W^\varepsilon_z \right\rangle \right], \end{aligned} \tag{87}$$

where $G^{(2)}_\theta$ denotes the operator

$$G^{(2)}_\theta \phi \equiv \int \mathcal{Q}_2(\theta \otimes \theta)(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \phi(\mathbf{y}, \mathbf{q}) \, d\mathbf{y} d\mathbf{q}.$$

Similarly

$$\begin{aligned} R^\varepsilon_3(z) = {} & \varepsilon^2 \tilde{k}^2 f'_z \left[\frac{1}{\tilde{k}} \left\langle W^\varepsilon_z, \mathbf{p} \cdot \nabla_{\mathbf{x}} (\tilde{\mathcal{L}}^\varepsilon_z \tilde{\mathcal{L}}^\varepsilon_z \theta) \right\rangle + \frac{\tilde{k}}{\varepsilon} \left\langle W^\varepsilon_z, \mathcal{L}^\varepsilon_z \tilde{\mathcal{L}}^\varepsilon_z \tilde{\mathcal{L}}^\varepsilon_z \theta \right\rangle \right] \\ & + \varepsilon^2 \frac{\tilde{k}^2}{2} f''_z \left[\frac{1}{\tilde{k}} \left\langle W^\varepsilon_z, \mathbf{p} \cdot \nabla_{\mathbf{x}} \theta \right\rangle + \frac{\tilde{k}}{\varepsilon} \left\langle W^\varepsilon_z, \mathcal{L}^\varepsilon_z \theta \right\rangle \right] \left[\left\langle W^\varepsilon_z, \tilde{\mathcal{L}}^\varepsilon_z \tilde{\mathcal{L}}^\varepsilon_z \theta \right\rangle - A^{(2)}_1(W^\varepsilon_z) \right] \\ & - \varepsilon^2 \tilde{k}^2 f'_z \left[\frac{1}{\tilde{k}} \left\langle W^\varepsilon_z, \mathbf{p} \cdot \nabla_{\mathbf{x}} (\mathcal{Q}'_2 \theta) \right\rangle + \frac{\tilde{k}}{\varepsilon} \left\langle W^\varepsilon_z, \mathcal{L}^\varepsilon_z \mathcal{Q}'_2 \theta \right\rangle \right]. \end{aligned} \tag{88}$$

Proposition 4.

$$\lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E} |R^\varepsilon_2(z)| = 0, \quad \lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E} |R^\varepsilon_3(z)| = 0.$$

Proof. Part of the argument is analogous to that given for Proposition 3. The additional estimates that we need to consider are the following:

In R^ε_2 (87):

$$\begin{aligned} & \sup_{z < z_0} \varepsilon^2 \mathbb{E} \left| \left\langle W^\varepsilon_z, \mathbf{p} \cdot \nabla_{\mathbf{x}} (G^{(2)}_\theta W^\varepsilon_z) \right\rangle \right| \\ & \le c \varepsilon^2 \gamma^{-2} \|W_0\|_2 \mathbb{E} \left\{ \left\| \nabla_{\mathbf{y}} \cdot \nabla_{\mathbf{x}} \mathcal{F}^{-1}_2 \theta(\mathbf{x}, \mathbf{y}) \right. \right. \\ & \quad \left. \left. \times \int \mathbb{E} \left[\delta_\gamma \tilde{V}^\varepsilon_z(\mathbf{x}, \mathbf{y}) \delta_\gamma \tilde{V}^\varepsilon_z(\mathbf{x}', \mathbf{y}') \right] \mathcal{F}^{-1}_2 \theta(\mathbf{x}', \mathbf{y}') \mathcal{F}^{-1}_2 W^\varepsilon_z(\mathbf{x}', \mathbf{y}') d\mathbf{x}' d\mathbf{y}' \right\|_2 \right\} \\ & \le c \varepsilon^2 \gamma^{-2} \|W_0\|_2 \mathbb{E} \left\{ \left\| \nabla_{\mathbf{y}} \cdot \nabla_{\mathbf{x}} \mathcal{F}^{-1}_2 \theta(\mathbf{x}, \mathbf{y}) \mathbb{E} \left[\delta_\gamma \tilde{V}^\varepsilon_z(\mathbf{x}, \mathbf{y}) \right]^2 \right. \right. \\ & \quad \left. \left. \times \int \left| \mathcal{F}^{-1}_2 \theta(\mathbf{x}', \mathbf{y}') \mathcal{F}^{-1}_2 W^\varepsilon_z(\mathbf{x}', \mathbf{y}') \right| d\mathbf{x}' d\mathbf{y}' \right\|_2 \right\} \end{aligned}$$

$$
\begin{aligned}
&\le c\varepsilon^2\gamma^{-2}\|W_0\|_2 \left\| \nabla_{\mathbf{y}} \cdot \nabla_{\mathbf{x}} \mathcal{F}_2^{-1}\theta \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2 \right\|_2 \mathbb{E}\left\| \mathcal{F}_2^{-1}\theta \mathcal{F}_2^{-1} W_z^\varepsilon \right\|_2 \\
&\le c\varepsilon^2\gamma^{-2}\|\theta\|_2\|W_0\|_2^2 \left\| \nabla_{\mathbf{y}} \cdot \nabla_{\mathbf{x}} \mathcal{F}_2^{-1}\theta \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2 \right\|_2 \\
&\le c\|\theta\|_2\|W_0\|_2^2\varepsilon^2\gamma^{-1} \left\| [\mathcal{F}_2^{-1}\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}}\theta](\mathbf{x},\mathbf{y}) \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2(\mathbf{y}) \right\|_2 \\
&\quad + ac\|\theta\|_2\|W_0\|_2^2\varepsilon^2\gamma^{-2} \left\| [\mathcal{F}_2^{-1}\nabla_{\mathbf{x}}\theta](\mathbf{x},\mathbf{y}) \cdot \nabla_{\mathbf{y}} \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2(\mathbf{y}) \right\|_2 \\
&\le c\|\theta\|_2\|W_0\|_2^2\varepsilon^2\gamma^{-1} \sup_{|\mathbf{y}|\le L} \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2(\mathbf{y}) \\
&\quad + c\|\theta\|_2\|W_0\|_2^2\varepsilon^2\gamma^{-2} \sup_{|\mathbf{y}|\le L} \left| \nabla_{\mathbf{y}} \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2(\mathbf{y}) \right| \\
&\le O\left(\varepsilon^2\eta^{-2}\gamma|\min(\rho,\gamma^{-1})|^{2-2H} + \varepsilon^2\eta^{-2}\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H}\right)
\end{aligned}
$$

by Lemma 2 where L is the radius of the ball containing the support of θ. Further delineation yields the following order-of-magnitude estimates:

$$
\begin{cases}
\varepsilon^2 & \text{if } \eta, \rho \text{ held fixed} \\
\varepsilon^2\rho^{1-H} & \text{if } \eta, \gamma \text{ held fixed} \\
\varepsilon^2\eta^{-2}\rho^{1-H} & \text{if } \gamma \text{ held fixed} \\
\varepsilon^2\eta^{-2} & \text{if } \rho \text{ held fixed} \\
\varepsilon^2\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H} & \text{if } \eta \text{ held fixed.}
\end{cases}
$$

Consider the next term:

$$
\begin{aligned}
&\sup_{z<z_0} \varepsilon\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon G_\theta^{(2)} W_z^\varepsilon \right\rangle\right| \\
&\le c\varepsilon^2\gamma^{-3}\|W_0\|_2 \mathbb{E}\left\{ \left\| \delta_\gamma V_z^\varepsilon(\mathbf{x},\mathbf{y}) \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \right.\right. \\
&\quad \left.\left. \times \int \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y}) \delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x}',\mathbf{y}')\right] \mathcal{F}_2^{-1}\theta(\mathbf{x}',\mathbf{y}') \mathcal{F}_2^{-1} W_z^\varepsilon(\mathbf{x}',\mathbf{y}') d\mathbf{x}' d\mathbf{y}' \right\|_2 \right\} \\
&\le c\varepsilon^2\gamma^{-3}\|W_0\|_2 \mathbb{E}\left\{ \left\| \delta_\gamma V_z^\varepsilon(\mathbf{x},\mathbf{y}) \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})\right]^2 \right.\right. \\
&\quad \left.\left. \times \int \left| \mathcal{F}_2^{-1}\theta(\mathbf{x}',\mathbf{y}') \mathcal{F}_2^{-1} W_z^\varepsilon(\mathbf{x}',\mathbf{y}') \right| d\mathbf{x}' d\mathbf{y}' \right\|_2 \right\} \\
&\le c\varepsilon^2\gamma^{-3}\|\theta\|_2\|W_0\|_2^2 \mathbb{E} \left\| \delta_\gamma V_z^\varepsilon(\mathbf{x},\mathbf{y}) \mathcal{F}_2^{-1}\theta \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2 \right\|_2 \\
&\le O\left(\varepsilon^2\eta^{-2}|\min(\rho,\gamma^{-1})|^{3-3H}\right)
\end{aligned}
$$

by Corollary 3.

In R_3^ε (88):

$$\sup_{z<z_0} \varepsilon \mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\right\rangle\right| \le \varepsilon \|W_0\|_2 \sup_{z<z_0} \sqrt{\mathbb{E}\left\|\mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\right\|_2^2}$$
$$= O\left(\varepsilon \gamma^{-3} \sup_{|\mathbf{y}|\le L} \mathbb{E}\left|\delta_\gamma \tilde{V}_z^\varepsilon\right|^2(\mathbf{y}) \mathbb{E}^{1/2}\left|\delta_\gamma V_z^\varepsilon\right|^2(\mathbf{y})\right)$$
$$= O\left(\varepsilon \eta^{-2} |\min(\rho, \gamma^{-1})|^{3-3H}\right),$$

by (62) and Lemma 2. The preceding two terms can be estimated from above by the following order of magnitude:

$$\begin{cases} \varepsilon & \text{if } \rho \text{ and } \eta \text{ held fixed} \\ \varepsilon & \text{if } \gamma \text{ and } \eta \text{ held fixed} \\ \varepsilon\eta^{-2} & \text{if } \gamma \text{ or } \rho \text{ held fixed} \\ \varepsilon |\min(\rho, \gamma^{-1})|^{3-3H} & \text{if } \eta \text{ held fixed;} \end{cases}$$

$$\varepsilon^2 \mathbb{E}\left|\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\mathcal{Q}_2'\theta)\right\rangle\right| \le \varepsilon^2 \sqrt{\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\mathcal{Q}_2'\theta)\right\rangle\right|^2}$$
$$\le c\varepsilon^2\gamma^{-2}\|W_0\|_2 \left\|\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{x}}\mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})\right]^2 \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\right\|_2$$
$$= O\left(\varepsilon^2\gamma^{-2}\mathbb{E}_{|\mathbf{y}|\le L}\left|\nabla_{\mathbf{y}}\mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right|\right)$$
$$= O\left(\varepsilon^2\eta^{-2}\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H}\right) \tag{89}$$

which in the various regimes has the following order of magnitude:

$$\begin{cases} \varepsilon^2 & \text{if } \rho \text{ and } \eta \text{ held fixed} \\ \varepsilon^2\rho^{1-H} & \text{if } \gamma \text{ and } \eta \text{ held fixed} \\ \varepsilon^2\eta^{-2}\rho^{1-H} & \text{if } \gamma \text{ held fixed} \\ \varepsilon^2\eta^{-2} & \text{if } \rho \text{ held fixed} \\ \varepsilon^2\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H} & \text{if } \eta \text{ held fixed;} \end{cases}$$

$$\varepsilon \mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \mathcal{Q}_2'\theta\right\rangle\right| \le \varepsilon\sqrt{\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \mathcal{Q}_2'\theta\right\rangle\right|^2}$$
$$\le c\varepsilon^2\gamma^{-3}\|W_0\|_2\mathbb{E}\left\|\delta_\gamma V_z^\varepsilon(\mathbf{x},\mathbf{y})\mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})\right]^2\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\right\|_2$$
$$= O\left(\varepsilon^2\gamma^{-3}\sup_{|\mathbf{y}|\le L}\mathbb{E}\left|\delta_\gamma \tilde{V}_z^\varepsilon\right|^2(\mathbf{y})\mathbb{E}^{1/2}\left|\delta_\gamma V_z^\varepsilon\right|^2(\mathbf{y})\right)$$
$$= O\left(\varepsilon^2\eta^{-2}|\min(\rho,\gamma^{-1})|^{3-3H}\right) \tag{90}$$

by Lemma 2. □

Consider the test function $f_z^\varepsilon = f_z + f_{1,z}^\varepsilon + f_{2,z}^\varepsilon + f_{3,z}^\varepsilon$. We have

$$\mathcal{A}^\varepsilon f_z^\varepsilon = \frac{1}{\tilde{k}} f_z' \left\langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_{\mathbf{x}} \theta \right\rangle + \tilde{k}^2 f_z'' A_2^{(1)}(W_z^\varepsilon) + \tilde{k}^2 f' A_1^{(1)}(W_z^\varepsilon) + R_2^\varepsilon(z) + R_3^\varepsilon(z) + R_1^\varepsilon(z). \tag{91}$$

Set

$$R^\varepsilon(z) = R_1^\varepsilon(z) + R_2^\varepsilon(z) + R_3^\varepsilon(z). \tag{92}$$

It follows from Propositions 2 and 4 that

$$\lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E}|R^\varepsilon(z)| = 0.$$

For the tightness it remains to show

Proposition 5. $\{\mathcal{A}^\varepsilon f_z^\varepsilon\}$ *are uniformly integrable.*

Proof. Indeed, each term in the expression (91) is uniformly integrable. We only need to be concerned with terms in $R^\varepsilon(z)$ since other terms are obviously uniformly integrable because W_z^ε is uniformly bounded in the square norm. But since the previous estimates establish the uniform boundedness of the second moments of the corresponding terms, the uniform integrability of the terms follow. □

4.2. Identification of the limit. Our strategy is to show directly that in passing to the weak limit the limiting process solves the martingale problem formulated in Sect. 2.1. The uniqueness of the martingale solution mentioned in Sect. 2.4 then identifies the limiting process as the unique $L^2(\mathbb{R}^{2d})$-valued solution to the initial value problem of the stochastic PDE (38).

Recall that for any C^2-function f,

$$\begin{aligned} M_z^\varepsilon(\theta) &= f_z^\varepsilon - \int_0^z \mathcal{A}^\varepsilon f_s^\varepsilon \, ds \\ &= f_z + f_1^\varepsilon(z) + f_2^\varepsilon(z) + f_3^\varepsilon(z) - \int_0^z \frac{1}{\tilde{k}} f_z' \left\langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_{\mathbf{x}} \theta \right\rangle ds \\ &\quad - \int_0^z \tilde{k}^2 \left[f_s'' A_2^{(1)}(W_s^\varepsilon) + f_s' A_1^{(1)}(W_s^\varepsilon) \right] ds - \int_0^z R^\varepsilon(s) \, ds \end{aligned} \tag{93}$$

is a martingale. The martingale property implies that for any finite sequence $0 < z_1 < z_2 < z_3 < \ldots < z_n \le z$, C^2-function f and bounded continuous function h with compact support, we have

$$\mathbb{E}\left\{ h\left(\left\langle W_{z_1}^\varepsilon, \theta \right\rangle, \left\langle W_{z_2}^\varepsilon, \theta \right\rangle, \ldots, \left\langle W_{z_n}^\varepsilon, \theta \right\rangle \right) \left[M_{z+s}^\varepsilon(\theta) - M_z^\varepsilon(\theta) \right] \right\} = 0, \quad \forall s > 0, \quad z_1 \le z_2 \le \cdots \le z_n \le z. \tag{94}$$

Let

$$\bar{\mathcal{A}} f_z \equiv f_s' \left[\frac{1}{\tilde{k}} \langle W_z, \mathbf{p} \cdot \nabla_{\mathbf{x}} \theta \rangle + \tilde{k}^2 \bar{A}_1(W_z) \right] + \tilde{k}^2 f_z'' \bar{A}_2(W_z),$$

where

$$\bar{A}_2(\phi) = \lim_{\rho\to\infty} A_2^{(1)}(\phi) = \int \overline{Q}(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\phi(\mathbf{x},\mathbf{p})\phi(\mathbf{y},\mathbf{q})d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{q}, \quad (95)$$

$$\bar{A}_1(\phi) = \lim_{\rho\to\infty} A_1^{(1)}(\theta) = \int \overline{Q}_0(\theta)(\mathbf{x},\mathbf{p})\phi(\mathbf{x},\mathbf{p})d\mathbf{x}d\mathbf{p}, \quad (96)$$

where $\overline{Q}(\theta\otimes\theta)$ and $\overline{Q}_0(\theta)$ are given by (33) and (32), respectively. For $\rho\to\infty, \gamma\to 0$ as $\varepsilon\to 0$ the limits in (95) are not well-defined unless $H\in(0,1/2)$ in the worst case scenario allowed by (7). Likewise, the convergence does not hold for $H\in[1/2,1)$ when $\eta\to 0$ in the worst case scenario allowed by (7).

For each possible limit process in $D([0,\infty); L^2_w(\mathbb{R}^{2d}))$ there is at most a countable set of discontinuous points with a positive probability and we consider all the finite set $\{z_1,...,z_n\}$ in (94) to be outside of the set of discontinuity.

In view of the results of Propositions 1, 2, 3, 4 we see that f_z^ε and $\mathcal{A}^\varepsilon f_z^\varepsilon$ in (93) can be replaced by f_z and $\bar{\mathcal{A}} f_z$, respectively, modulo an error that vanishes as $\varepsilon\to 0$. With this and the tightness of $\{W_z^\varepsilon\}$ we can pass to the limit $\varepsilon\to 0$ in (94). We see that the limiting process satisfies the martingale property that

$$\mathbb{E}\left\{h\left(\langle W_{z_1},\theta\rangle, \langle W_{z_2},\theta\rangle,...,\langle W_{z_n},\theta\rangle\right)\left[M_{z+s}(\theta)-M_z(\theta)\right]\right\}=0, \quad \forall s>0,$$

where

$$M_z(\theta) = f_z - \int_0^z \bar{\mathcal{A}} f_s\,ds. \quad (97)$$

Then it follows that

$$\mathbb{E}\left[M_{z+s}(\theta)-M_z(\theta)|W_u, u\le z\right]=0, \quad \forall z,s>0$$

which proves that $M_z(\theta)$ is a martingale.

Note that $\langle W_z^\varepsilon,\theta\rangle$ is uniformly bounded:

$$\left|\langle W_z^\varepsilon,\theta\rangle\right| \le \|W_0\|_2\|\theta\|_2$$

so we have the convergence of the second moment

$$\lim_{\varepsilon\to 0}\mathbb{E}\left\{\langle W_z^\varepsilon,\theta\rangle^2\right\} = \mathbb{E}\left\{\langle W_z,\theta\rangle^2\right\}.$$

Using $f(r)=r$ and r^2 in (97) we see that

$$M_z^{(1)}(\theta) = \langle W_z,\theta\rangle - \int_0^z\left[\frac{1}{\tilde{k}}\langle W_s,\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\rangle - \tilde{k}^2\bar{A}_3(W_s)\right]ds$$

is a martingale with the quadratic variation

$$\left[M^{(1)}(\theta), M^{(1)}(\theta)\right]_z = \tilde{k}^2\int_0^z \bar{A}_2(W_s)\,ds = \tilde{k}^2\int_0^z\left\langle W_s,\overline{\mathcal{K}}_\theta W_s\right\rangle ds,$$

where $\overline{\mathcal{K}}_\theta$ is defined as in (31).

Appendix A. Mixing Coefficients and Moment Estimates for $\tilde{V}_z$

Let $\mathcal{F}_z$ and $\mathcal{F}_z^+$ be the sigma-algebras generated by $\{V_s : \forall s \le z\}$ and $\{V_s : \forall s \ge z\}$, respectively.

Consider the strong mixing coefficient

$$\begin{aligned}\alpha(t) &= \sup_{A \in \mathcal{F}_{z+t}^+} \sup_{B \in \mathcal{F}_z} |P(AB) - P(A)P(B)| \\ &= \frac{1}{2} \sup_{A \in \mathcal{F}_{z+t}^+} \mathbb{E}\left[|P(A|\mathcal{F}_z) - P(A)|\right]\end{aligned}$$

which can be used to bound the first order moment:

$$\mathbb{E}\left[|\mathbb{E}\left[V_s|\mathcal{F}_z\right]|\right] \le 8\alpha(s-z)^{1/p}\left[\mathbb{E}|V_s|^q\right]^{1/q}, \quad \forall s > z, \quad p^{-1} + q^{-1} = 1$$

([9], Cor. 2.4). Hence the integrability of $\alpha(t)$ implies that $\tilde{V}_z$ has a finite first order moment.

To bound the higher order moments of $\tilde{V}_z$ one can consider, for example, the general L^p-mixing coefficients

$$\begin{aligned}\phi_p(t) &= \sup_{A \in \mathcal{F}_{z+t}} \mathbb{E}^{1/p}\left[|P(A|\mathcal{F}_z) - P(A)|^p\right], \quad p \in [1, \infty) \\ &= \sup_{h \in L^p(P, \mathcal{F}_{z+t}^+)} \sup_{\substack{g \in L^q(P, \mathcal{F}_z) \\ \mathbb{E}g^q = 1, \mathbb{E}g = 0}} \mathbb{E}[hg], \quad p^{-1} + q^{-1} = 1, \quad p \in [1, \infty).\end{aligned}$$

We note that $\alpha(t) = \phi_1(t)$ and for $p = \infty$,

$$\begin{aligned}\phi_\infty(t) &= \sup_{A \in \mathcal{F}_{t+z}^+} \sup_{\substack{B \in \mathcal{F}_z \\ P(B) > 0}} |P(A|B) - P(A)|, \quad \forall t \ge 0 \\ &= \sup_{A \in \mathcal{F}_{t+z}^+} \text{ess-sup}_\omega |P(A|\mathcal{F}_z) - P(A)| \\ &\equiv \phi(t)\end{aligned}$$

is called the uniform mixing coefficient [9]. In terms of ϕ_p one has the following estimate

$$|\mathbb{E}[h_1 h_2] - \mathbb{E}[h_1]\mathbb{E}[h_2]| \le 2^{\min(q,2)} \phi_p(t)^{1/u} \mathbb{E}^{1/(vp)}[h_2^{vp}] \mathbb{E}^{1/q}[h_1^q] \tag{98}$$

for $u, v, p, q \in [1, \infty]$, $u^{-1} + v^{-1} = 1$, $p^{-1} + q^{-1} = 1$ and real-valued $h_1 \in L^q(\Omega, \mathcal{F}_z, P)$, $h_2 \in L^{vp}(\Omega, \mathcal{F}_{z+t}^+, P)$ (see [9], Prop. 2.2). In particular, for $q > 2$, $v = q/p$,

$$|\mathbb{E}[h_1 h_2] - \mathbb{E}[h_1]\mathbb{E}[h_2]| \le 4\phi_p(t)^{(q-p)/q} \mathbb{E}^{1/q}[h_2^q] \mathbb{E}^{1/q}[h_1^q], \quad p^{-1} + q^{-1} = 1 \tag{99}$$

by which, along with the Hölder inequality, we can bound the second moment of $\tilde{V}_z$ as follows: First we observe that for $s, \tau \ge z$ and $h_1 = \mathbb{E}_z(V_s)$, $h_2 = V_\tau$,

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}_z[V_s(\mathbf{x})]\mathbb{E}_z[V_\tau(\mathbf{x})]\right] \\ &\quad = \mathbb{E}\left[\mathbb{E}_z[V_s(\mathbf{x})]V_\tau(\mathbf{x})\right] \le 4\phi_p(\tau - z)^{(q-p)/q} \mathbb{E}^{1/q}[V_z^q] \mathbb{E}^{1/q}[\mathbb{E}_z^q[V_s]].\end{aligned}$$

By setting $s = \tau$ first and the Cauchy-Schwartz inequality we have

$$\mathbb{E}\left[\mathbb{E}_z^2[V_s]\right] \le 4\phi_p(s-z)^{(q-p)/q}\mathbb{E}^{2/q}[V_z^q],$$
$$\mathbb{E}\left[\mathbb{E}_z[V_s(\mathbf{x})]\mathbb{E}_z[V_\tau(\mathbf{x})]\right] \le 4\phi_p(s-z)^{(q-p)/(2q)}\phi_p \times(\tau-z)^{(q-p)/(2q)}\mathbb{E}^{2/q}[V_z^q], \quad s,\tau \ge z.$$

Hence

$$\begin{aligned}\mathbb{E}[\tilde{V}_z^2] &\le 2\int_z^\infty\int_z^\infty \mathbb{E}\left[\mathbb{E}_z[V_\tau]\mathbb{E}_z[V_s]\right]dsd\tau + 2\int_0^\infty\int_0^\infty \mathbb{E}\left[\mathbb{E}_0[V_\tau]\mathbb{E}_0[V_s]\right]dsd\tau \\ &\le 8\mathbb{E}^{2/q}[V_z^q]\left(\int_0^\infty \phi_p(t)^{(q-p)/(2q)}dt\right)^2 \\ &\le 8\mathbb{E}^{1/3}[V_z^6]\left(\int_0^\infty \phi_{6/5}^{2/5}(t)dt\right)^2\end{aligned}$$

which is finite if $\phi_{6/5}^{2/5}(t)$ is integrable (if V_z is assumed to have a finite 6^{th} order moment).

When V_z is almost surely bounded, the preceding calculation with $p = 1, q = \infty$ becomes

$$\mathbb{E}[\tilde{V}_z^2] \le 8\lim_{q\to\infty}\mathbb{E}^{1/q}[V_z^q]\left(\int_0^\infty \phi_1^{1/2}(t)dt\right)^2$$

which is finite when $\phi_1^{1/2}(t)$ is integrable.

In order to bound higher order moments in the non-Gaussian case, one can assume the integrability of the uniform mixing coefficient $\phi(t) \equiv \phi_\infty(t)$. Then we have

$$|P(A|\mathcal{F}_z) - P(A)| \le \phi(s-z), \quad \forall A \in \mathcal{F}_s, \quad s \ge z,$$

and for $p \in [1,\infty)$, $p^{-1} + q^{-1} = 1$,

$$\left|\mathbb{E}\left[V_s|\mathcal{F}_z\right]\right| \le 2^{1/p}\phi^{1/p}(s-z)\left[\mathbb{E}\left[V_s^q|\mathcal{F}_z\right] + \mathbb{E}\left[V_s^q\right]\right]^{1/q} \tag{100}$$

(cf. [9], Prop. 2.6). Using (100) and the Hölder inequality repeatedly we obtain

$$\mathbb{E}\left\{\int_z^\infty \mathbb{E}\left[V_s|\mathcal{F}_z\right]ds\right\}^p \le c\left[\int_0^\infty \phi(s)ds\right]^p \mathbb{E}[V_s^p]. \tag{101}$$

Hence the integrability of $\phi(t)$ implies that $\tilde{V}_z$ given by (51) has a finite moment of any order $p < \infty$ if V_z has a finite moment of order p.

In summary we have

Proposition 1. (i) *Assume that* $\mathbb{E}[V_z^p] < \infty$, $p \in [1,\infty)$. *If the uniform (L^∞-) mixing coefficient $\phi_\infty(t)$ of V_z is integrable then $\tilde{V}_z$ has finite moments of order p.*
(ii) *Assume that* $\mathbb{E}[V_z^6] < \infty$. *If the 2/5-power of the $L^{6/5}$-mixing coefficient $\phi_{6/5}(t)$ is integrable, then $\tilde{V}_z$ has finite second moment.*
(iii) *Assume V_z is almost surely bounded. If the square-root of the alpha- (L^1-) mixing coefficient $\phi_1(t)$ is integrable then $\tilde{V}_z$ has finite second moment.*

Appendix B. Proof of Lemma 2

(i) Estimation of $\sup_{\substack{|z|\le z_0\\|\mathbf{y}|\le L}} \mathbb{E}\left[\left(\delta_\gamma V_z^\varepsilon\right)^2\right](\mathbf{y})$: We have that for $\gamma\rho \le 1$,

$$\begin{aligned}
&\sup_{|z|\le z_0} \mathbb{E}\left[\left(\delta_\gamma V_z^\varepsilon(\mathbf{x},\mathbf{y})\right)^2\right]\\
&= \sup_{|z|\le z_0} \int 4|\sin(\gamma\mathbf{y}\cdot\mathbf{k}/2)|^2 \Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k}\\
&\le \sup_{|z|\le z_0} \int |\gamma\mathbf{y}\cdot\mathbf{k}|^2\, \Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k}\\
&\le c_0\gamma^2|\mathbf{y}|^2 \sup_{|z|\le z_0} \int_{|\xi|\le\rho}\int_{|\mathbf{k}|\le\rho} (\eta^2+|\mathbf{k}|^2+|\xi|^2)^{-H-(d+1)/2}|\mathbf{k}|^{d+1}d|\mathbf{k}|d\xi\\
&\le c_1\gamma^2|\mathbf{y}|^2 \sup_{|z|\le z_0} \int_{|\xi|\le\rho} (\eta^2+|\xi|^2)^{-H+1/2}d\xi\\
&\le c_2\gamma^2|\mathbf{y}|^2 \int_{|\xi|\in(\eta,\rho)} |\xi|^{-2H+1}d\xi\\
&\le c_3\gamma^2|\mathbf{y}|^2\left(\eta^{2-2H}+\rho^{2-2H}\right).
\end{aligned}$$

For $\rho\gamma \ge 1$ we divide the domain of integration into $I_0 = \{|\mathbf{k}| \le \gamma^{-1}\}$ and $I_1 = \{|\mathbf{k}| \ge \gamma^{-1}\}$ and estimate their contributions separately. For I_0 the upper bound is similar to the above, namely, we have

$$\sup_{|z|\le z_0} \int_{I_0} 4|\sin(\gamma\mathbf{y}\cdot\mathbf{k}/2)|^2 \Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k} \le c_4\gamma^2|\mathbf{y}|^2\left(\eta^{2-2H}+\gamma^{-2+2H}\right).$$

For I_1 we have instead that

$$\begin{aligned}
&\sup_{|z|\le z_0} \int_{I_1} 4|\sin(\gamma\mathbf{y}\cdot\mathbf{k}/2)|^2 \Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k}\\
&\le 4\sup_{|z|\le z_0} \int_{I_1} \Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k}\\
&\le c_5 \sup_{|z|\le z_0} \int_{|\xi|\in(\gamma^{-1},\rho)} (\eta^2+|\xi|^2)^{-H-1/2}d\xi\\
&\le c_6 \int_{|\xi|\in(\gamma^{-1},\rho)} |\xi|^{-2H-1}d\xi\\
&\le c_7\left(\gamma^{2H}+\rho^{-2H}\right).
\end{aligned}$$

Put together, the upper bound becomes

$$\sup_{\substack{|z|\le z_0\\|\mathbf{x}|,|\mathbf{y}|\le L}} \mathbb{E}\left[\left(\delta_\gamma V_z^\varepsilon(\mathbf{x},\mathbf{y})\right)^2\right] \le c_8\gamma^2\left|\min(\gamma^{-1},\rho)\right|^{2-2H}, \quad \gamma,\eta\le 1\le\rho.$$

(ii) Estimation of $\sup_{|z|\le z_0}\mathbb{E}\left[\tilde{V}^\varepsilon_z(\mathbf{x})\right]^2$: It follows from the argument for Corollary 1 and Assumption 2 that

$$\begin{aligned}\mathbb{E}\left[\tilde{V}^\varepsilon_z(\mathbf{x})\right]^2 &\le \left(\int_0^\infty r_{\eta,\rho}(t)dt\right)^2 \mathbb{E}[V^\varepsilon_z]^2\\ &\le c\eta^{-2}\eta^{-2H}.\end{aligned}$$

(iii) Estimation of $\sup_{\substack{|z|\le z_0\\ |\mathbf{y}|\le L}}\mathbb{E}\left[\left(\delta_\gamma\tilde{V}^\varepsilon_z\right)^2\right](\mathbf{y})$: First note that the correlation coefficient for $\delta_\gamma\tilde{V}^\varepsilon_z$ is bounded from above by $cr_{\eta,\rho}(t)$ for some constant $c>0$. Then we have as in (i), (ii) that

$$\begin{aligned}\mathbb{E}\left[\delta_\gamma\tilde{V}^\varepsilon_z(\mathbf{x})\right]^2 &\le c_1\left(\int_0^\infty r_{\eta,\rho}(t)dt\right)^2 \mathbb{E}[\delta_\gamma V^\varepsilon_z]^2\\ &\le c_2\eta^{-2}\gamma^2\left|\min(\gamma^{-1},\rho)\right|^{2-2H}.\end{aligned}$$

(iv) Estimation of $\sup_{\substack{|z|\le z_0\\ |\mathbf{y}|\le L}}\left|\nabla_\mathbf{y}\mathbb{E}\left[\delta_\gamma\tilde{V}^\varepsilon_z\right]^2(\mathbf{y})\right|$: By the Cauchy-Schwartz inequality and the preceding calculation we have

$$\begin{aligned}&\sup_{\substack{|z|\le z_0\\ |\mathbf{y}|\le L}}\left|\nabla_\mathbf{y}\mathbb{E}\left[\delta_\gamma\tilde{V}^\varepsilon_z\right]^2(\mathbf{y})\right|\\ &\le c_1\sqrt{\gamma^2\mathbb{E}\left[\nabla_\mathbf{x}\tilde{V}^\varepsilon(\mathbf{x}+\gamma\mathbf{y}/2)+\nabla_\mathbf{x}\tilde{V}^\varepsilon(\mathbf{x}-\gamma\mathbf{y}/2)\right]^2}\sqrt{\mathbb{E}\left[\delta_\gamma\tilde{V}^\varepsilon(\mathbf{x},\mathbf{y})\right]^2}\\ &\le c_3\left(\int_0^\infty r_{\eta,\rho}(t)dt\right)^2\gamma\mathbb{E}^{1/2}\left[\nabla_\mathbf{x}V^\varepsilon\right]^2\mathbb{E}^{1/2}\left[\delta_\gamma V^\varepsilon_z(\mathbf{x},\mathbf{y})\right]^2\\ &\le c_4\eta^{-2}\gamma^2\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H}.\end{aligned}$$

(v) Estimation of $\sup_{|z|\le z_0}\mathbb{E}\|\mathbf{p}\cdot\nabla_\mathbf{x}(\tilde{\mathcal{L}}^\varepsilon_z\theta)\|_2^2$: A similar line of reasoning and a straightforward spectral calculation yield that

$$\begin{aligned}\mathbb{E}\|\mathbf{p}\cdot\nabla_\mathbf{x}(\tilde{\mathcal{L}}^\varepsilon_z\theta)\|_2^2 &= \mathbb{E}\|\nabla_\mathbf{y}\cdot\nabla_\mathbf{x}\gamma^{-1}\delta_\gamma\tilde{V}^\varepsilon_z\mathcal{F}_2^{-1}\theta\|_2^2\\ &\le c_1\mathbb{E}\|\nabla^2_\mathbf{x}\tilde{V}^\varepsilon_z\mathcal{F}_2^{-1}\theta\|_2^2\\ &\le c_2\eta^{-2}\mathbb{E}\left[\nabla^2_\mathbf{x}V^\varepsilon_z\right]^2\\ &\le c_3\eta^{-2}\rho^{4-2H}.\end{aligned}$$

Appendix C. Proof of Corollary 3

By the Cauchy-Schwartz inequality we have the following calculation:

$$\begin{aligned}&\mathbb{E}\left[\|\mathcal{L}^\varepsilon_z\theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}^\varepsilon_z\theta(\mathbf{y},\mathbf{q})\|_2^2\right]\\ &\le C_1\left\{\left\|\mathbb{E}\left[\mathcal{L}^\varepsilon_z\theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}^\varepsilon_z\theta(\mathbf{y},\mathbf{q})\right]\right\|_2^2+\mathbb{E}\left[\|\mathcal{L}^\varepsilon_z\theta(\mathbf{x},\mathbf{p})\|_2^2\right]\mathbb{E}\left[\left\|\tilde{\mathcal{L}}^\varepsilon_z\theta(\mathbf{y},\mathbf{q})\right\|_2^2\right]\right\}\\ &= C_1\gamma^{-4}\left\{\left\|\mathbb{E}\left[\delta_\gamma V^\varepsilon_z(\mathbf{x},\mathbf{x}')\delta_\gamma\tilde{V}^\varepsilon_z(\mathbf{y},\mathbf{y}')\right]\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{x}')\mathcal{F}_2^{-1}\theta(\mathbf{y},\mathbf{y}')\right\|_2^2\right.\end{aligned}$$

$$+ \left\| \mathbb{E}\left[\left| \delta_\gamma V_z^\varepsilon \right|^2 \right] \mathcal{F}_2^{-1}\theta \right\|_2^2 \left\| \mathbb{E}\left[\left| \delta_\gamma \tilde{V}_z^\varepsilon \right|^2 \right] \mathcal{F}_2^{-1}\theta \right\|_2^2 \Bigg\}$$

$$= O\left(\sup_{|\mathbf{y}|\le L} \mathbb{E}\left|\delta_\gamma V_z^\varepsilon\right|^2(\mathbf{y}) \mathbb{E}\left|\delta_\gamma \tilde{V}_z^\varepsilon\right|^2(\mathbf{y}) \right)$$

and

$$\begin{aligned}
&\mathbb{E}\left[\|\mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\|_2^2 \right] \\
&\le C_1' \left\{ \gamma^{-4} \int \mathbb{E}\left[\delta_\gamma V_z^\varepsilon\right]^2 \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2 \left(\mathcal{F}_2^{-1}\theta\right)^2 d\mathbf{x}d\mathbf{y} + \left\| \mathbb{E}\left[\mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta(\mathbf{x},\mathbf{p}) \right] \right\|_2^2 \right\} \\
&= C_1' \gamma^{-4} \Bigg\{ \int \mathbb{E}\left[\delta_\gamma V_z^\varepsilon\right]^2 \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2 \left(\mathcal{F}_2^{-1}\theta\right)^2 d\mathbf{x}d\mathbf{y} \\
&\quad + \left\| \mathbb{E}\left[\delta_\gamma V_z^\varepsilon \delta_\gamma \tilde{V}_z^\varepsilon \right] \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \right\|_2^2 \Bigg\} \\
&= O\left(\sup_{|\mathbf{y}|\le L} \mathbb{E}\left|\delta_\gamma V_z^\varepsilon\right|^2(\mathbf{y}) \mathbb{E}\left|\delta_\gamma \tilde{V}_z^\varepsilon\right|^2(\mathbf{y}) \right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\|\tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\|_2^2 \right] \\
&\le C_2 \left\{ \gamma^{-4} \int \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2 \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2 \left(\mathcal{F}_2^{-1}\theta\right)^2 d\mathbf{x}d\mathbf{y} + \left\| \mathbb{E}\left[\tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta(\mathbf{x},\mathbf{p}) \right] \right\|_2^2 \right\} \\
&= C_2 \gamma^{-4} \left\{ \int \left(\mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon\right]^2 \right)^2 \left(\mathcal{F}_2^{-1}\theta\right)^2 d\mathbf{x}d\mathbf{y} + \left\| \mathbb{E}\left[\delta_\gamma \tilde{V}_z^\varepsilon \delta_\gamma \tilde{V}_z^\varepsilon \right] \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \right\|_2^2 \right\} \\
&= O\left(\sup_{|\mathbf{y}|\le L} \mathbb{E}^2 \left|\delta_\gamma \tilde{V}_z^\varepsilon\right|^2(\mathbf{y}) \right),
\end{aligned}$$

where C_1, C_1', C_2 are constants independent of ρ, η, γ and L is the radius of the ball containing the support of $\mathcal{F}_2^{-1}\theta$. Similarly we have that

$$\mathbb{E}\left\| \mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta \right\|_2^2 = O\left(\sup_{|\mathbf{y}|\le L} \mathbb{E}^2 \left|\delta_\gamma \tilde{V}_z^\varepsilon\right|^2 \mathbb{E}\left|\delta_\gamma V_z^\varepsilon\right|^2 \right).$$

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