A LETTERS JOURNAL EXPLORING
THE FRONTIERS OF PHYSICS

OFFPRINT

Space-frequency correlation of classical waves in disordered media: High-frequency and small-scale asymptotics

A. C. FANNJIANG

EPL, **80** (2007) 14005

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

October 2007

EPL, **80** (2007) 14005 www.epljournal.org
doi: 10.1209/0295-5075/80/14005

Space-frequency correlation of classical waves in disordered media: High-frequency and small-scale asymptotics

A. C. FANNJIANG

Department of Mathematics, University of California - Davis, CA 95616-8633, USA

received 26 March 2007; accepted in final form 13 August 2007
published online 11 September 2007

PACS 42.25.Dd – Wave propagation in random media
PACS 42.68.Ay – Propagation, transmission, attenuation, and radiative transfer
PACS 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion

Abstract – Two-frequency radiative transfer (2f-RT) theory is developed for geometrical optics in random media. The space-frequency correlation is described by the two-frequency Wigner distribution (2f-WD) which satisfies a closed-form equation, the two-frequency Wigner-Moyal equation. In the RT regime it is proved rigorously that 2f-WD satisfies a Fokker-Planck–like equation with complex-valued coefficients. By dimensional analysis 2f-RT equation yields the scaling behavior of three physical parameters: the spatial spread, the coherence length and the coherence bandwidth. The sub-transport mean-free-path behavior is obtained in a closed form by analytically solving a paraxial 2f-RT equation.

Introduction. – Correlation functions of fields arise naturally in the description of fluctuations and are ubiquitous objects in statistical physics. The most basic of those are the second-order correlations in the space-time or space-frequency domain; the two are equivalent to each other via the Fourier transform. When the field fluctuations can be described as a Gaussian stochastic process, all the correlation functions of the field can then be expressed in term of the second-order ones, by the use of the moment theorem for Gaussian processes. The second-order space-frequency correlation then emerges as an indispensable tool for studying fluctuations of fields and is equivalent to the *mutual coherence function* describing the field correlation at two space-time points [1].

Spatial and temporal structures of ultrawide-band high-frequency fields can be appreciably affected by small random changes of the medium parameters characteristic of almost all astro- and geophysical environments. An important step toward the analytical understanding of pulse propagation in multiply scattering media is then to derive the equation for the space-frequency correlation, obtain the qualitative information about its behavior and, if possible, find its (asymptotic) solutions. This problem has been extensively studied in the literature, see, *e.g.*, [2–6]. The main distinction of our approach from previous works is that our approach to space-frequency correlation is carried out in terms of the two-frequency Wigner distribution (2f-WD) for which we will derive rigorously equations of relatively simple form in the radiative transfer (RT) regime and obtain an exact solution for the small-scale behavior below the transport mean-free-path [1,7].

The standard (equal-time or -frequency) Wigner distribution (WD) is a quasi-probability density function in phase space and was first introduced by Wigner [8] in connection to quantum thermodynamics and later found wide-ranging applications in classical [9,10], as well as in quantum optics [1,11]. In classical optics, a main use of the Wigner distribution is connected to high-frequency asymptotic and radiative transfer, both of which can be most naturally worked out from the first principle in phase space (see the reviews [12,13] and references therein).

The main advantage of 2f-RT over the traditional equal-time radiative transfer theory is that it describes not just the energetic transport but also the two space-time point mutual coherence in the following way.

Let the scalar wave field U_j, $j=1,2$, of wave number $k_j, j=1,2$ be governed by the reduced wave equation

$$\Delta U_j(\mathbf{r}) + k_j^2(\nu + V(\mathbf{r}))U_j(\mathbf{r}) = 0, \quad \mathbf{r} \in \mathbb{R}^3, \quad j=1,2, \tag{1}$$

where ν and V are, respectively, the mean and fluctuation of the refractive index associated are assumed to be real-valued, corresponding to a lossless medium. For simplicity, we restrict our attention to dispersionless media (see [14] for discussion on dispersive media). Here and below the background wave speed is set to be unity so that $k_j=\omega_j$. Let $u(t_j,\mathbf{x}_j), j=1,2$ be the time-dependent wave field at two space-time points $(t_j,\mathbf{x}_j), j=1,2$. Let $\mathbf{x}=(\omega_1\mathbf{x}_1+\omega_2\mathbf{x}_2)/2$ and $\mathbf{y}=\omega_1\mathbf{x}_1-\omega_2\mathbf{x}_2$. Then we have

$$\langle u(t_1,\mathbf{x}_1)u^*(t_2,\mathbf{x}_2)\rangle = \int e^{i\mathbf{p}\cdot\mathbf{y}}e^{i(\omega_2t_2-\omega_1t_1)}\left\langle W(\mathbf{x},\mathbf{p};\omega_1,\omega_2)\right\rangle \mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\mathbf{p} \tag{2}$$

where $W(\mathbf{x},\mathbf{p};\omega_1,\omega_2)$ is the 2f-WD defined by

$$W(\mathbf{x},\mathbf{p};\omega_1,\omega_2)= \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1\left(\frac{\mathbf{x}}{\omega_1}+\frac{\mathbf{y}}{2\omega_1}\right)U_2^*\left(\frac{\mathbf{x}}{\omega_2}-\frac{\mathbf{y}}{2\omega_2}\right)\mathrm{d}\mathbf{y}= \left(\omega_1\omega_2\right)^3\int e^{i\mathbf{x}\cdot\mathbf{q}}\hat{U}_1\left(\omega_1\mathbf{p}+\frac{\omega_1\mathbf{q}}{2}\right)\hat{U}_2^*\left(\omega_2\mathbf{p}-\frac{\omega_2\mathbf{q}}{2}\right)\mathrm{d}\mathbf{q}.$$

Here and below $\langle\cdot\rangle$ denotes the ensemble average. For *temporally stationary* signals, wave fields of different frequencies are uncorrelated and only the equal-frequency WD is necessary to describe the two-time correlation. In comparison, the single-time correlations with $t_1=t_2=t$ gives rise to the expression

$$\langle u(t,\mathbf{x}_1)u^*(t,\mathbf{x}_2)\rangle = \int \mathrm{d}\omega'\mathrm{d}\mathbf{p}\ e^{i\mathbf{p}\cdot\mathbf{y}}e^{i\omega' t} \times\int\mathrm{d}\omega\left\langle W(\mathbf{x},\mathbf{p};\omega-\omega'/2,\omega+\omega'/2)\right\rangle$$

which is equivalent to the central-frequency-integrated 2f-WD. For ease of notation, we will drop the frequency arguments when writing the 2f-WD below.

Weak-coupling limit. – The radiative transfer regime sets in when the scale of medium fluctuation is much smaller than the propagation distance but is comparable or much larger than the wavelength. Based on the general principle of central limit theorem, RT corresponds to the scaling limit which replaces $\nu+V$ in eq. (1) with

$$\frac{1}{\theta^2\varepsilon^2}\left(\nu+\sqrt{\varepsilon}V\left(\frac{\mathbf{r}}{\varepsilon}\right)\right),\qquad \theta>0,\qquad \varepsilon\ll 1, \tag{3}$$

where $\varepsilon>0$ and $\theta^{-1}>0$ are, respectively, the ratio of the scale of medium fluctuation to the propagation distance and the wavelength. Thus, $\varepsilon\theta$ is the ratio of the wavelength to the propagation distance and as a result we rescale the wave number as $k\to k/(\varepsilon\theta)$, giving rise to the prefactor $(\theta\varepsilon)^{-2}$. This is the so-called weak-coupling (or disorder) limit in kinetic theory [15] under which the localization cannot take place.

We assume that $V(\mathbf{x})$ is an ergodic, mean-zero, statistically homogeneous random field. As a consequence, V admits the spectral representation $V(\mathbf{x})=\int e^{i\mathbf{x}\cdot\mathbf{p}}\hat{V}(\mathrm{d}\mathbf{p})$, where the spectral measure $\hat{V}$ satisfies $\langle\hat{V}(\mathrm{d}\mathbf{p})\hat{V}(\mathrm{d}\mathbf{q})\rangle=\delta(\mathbf{p}+\mathbf{q})\Phi(\mathbf{p})\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}$ with Φ the power spectral density. Since V is real-valued, $\Phi(\mathbf{p})$ is real-valued, non-negative and possesses the symmetry $\Phi(\mathbf{p})=\Phi(-\mathbf{p}),\forall\mathbf{p}$.

Physically speaking radiative transfer belongs to the diffusive wave regime under the condition of a large dimensionless conductance $g\gg1$. Let A be the illuminated area, λ the wavelength of radiation and L the distance of propagation. Let $N_f=\lambda L/A$ be the inverse Fresnel number and ℓ_* the transport mean-free-path. The dimensionless conductance can then be expressed simply as $g=k\ell_*/N_f$. With the scaling (3), $k\ell_*\sim N_f^{-1}\sim\theta^{-1}\varepsilon^{-1}$ and hence $g\sim\theta^{-2}\varepsilon^{-2}\gg1$ for $\theta\varepsilon\ll1$.

To adapt to the weak coupling and the geometrical optics (see below) scalings we introduce the two parameters ε,θ into the 2f-WD and redefine it as

$$W^\varepsilon(\mathbf{x},\mathbf{p})= \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\theta\varepsilon\mathbf{y}}{2k_1}\right)U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\theta\varepsilon\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y} \tag{4}$$

In view of the definition, we see that both $\mathbf{x}$ and $\mathbf{p}$ are dimensionless. The particular scaling factors are introduced in (4) so that W^ε satisfies the following Wigner-Moyal equation *exactly* [14]:

$$\mathbf{p}\cdot\nabla W^\varepsilon=\frac{1}{\sqrt{\varepsilon}}\mathcal{L}W^\varepsilon, \tag{5}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}W^\varepsilon(\mathbf{x},\mathbf{p}) = \frac{i}{2\theta}\int\hat{V}(\mathrm{d}\mathbf{q})e^{i\frac{\mathbf{q}\cdot\mathbf{x}}{\varepsilon k_1}}W^\varepsilon\left(\mathbf{x},\mathbf{p}-\frac{\theta\mathbf{q}}{2k_1}\right) -\frac{i}{2\theta}\int\hat{V}(\mathrm{d}\mathbf{q})e^{i\frac{\mathbf{q}\cdot\mathbf{x}}{\varepsilon k_2}}W^\varepsilon\left(\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{2k_2}\right).$$

In contrast, the Sudarshan equations for the mutual coherence function are first order in time but nonlocal in space even in the case of free field [1].

High-frequency regime. – Before we consider the radiative transfer limit $\varepsilon\downarrow0$, further let us take the high-frequency limit $\theta\downarrow0$ while maintaining the following relationships:

$$\lim_{\theta\to0}k_1=\lim_{\theta\to0}k_2=k,\qquad \frac{k_2-k_1}{\theta\varepsilon k}=\beta, \tag{6}$$

where $\beta>0$ is independent of θ and ε, representing the normalized difference in wave number. Frequencies within the range described by (6) remain coherent with one another.

In this regime, we see from (4) that to leading order the center of two field points is $\mathbf{x}/k$ and the difference is $\theta\varepsilon(\mathbf{y}+\beta\mathbf{x})/k$. Passing to the limit $\theta\downarrow0$ in (5) we obtain the first-order partial differential equation

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}W^\varepsilon(\mathbf{x},\mathbf{p})= -\frac{1}{2k\sqrt{\varepsilon}}(\nabla V)\left(\frac{\mathbf{x}}{\varepsilon k}\right)\cdot[\nabla_{\mathbf{p}}-i\beta\mathbf{x}]W^\varepsilon(\mathbf{x},\mathbf{p}). \tag{7}$$

For $\beta=0$, eq. (7) is the static Liouville equation. For $\beta>0$, eq. (7) retains the wave character and is the focus of the subsequent analysis. We shall refer to eq. (7) as the two-frequency Liouville equation (2f-LE).

Consider, for instance, the WKB ansatz

$$U_j(\mathbf{r}) = A_j(\mathbf{r}) \exp\left(\frac{ik_j}{\theta\varepsilon} S_j(\mathbf{r})\right), \quad j = 1, 2,$$

where the phase S_j and the amplitude A_j depend on the frequency differentiably. In the first case, assume $S_1 = S_2 = S$. Then in the high-frequency limit 2f-WD becomes

$$W^\varepsilon(\mathbf{x}, \mathbf{p}) = e^{i\beta\mathbf{x}\cdot\mathbf{p}} e^{-i\beta k S(\frac{\mathbf{x}}{k})} |A|^2 \left(\frac{\mathbf{x}}{k}\right) \delta\left(\mathbf{p} - \nabla S\left(\frac{\mathbf{x}}{k}\right)\right) \quad (8)$$

which satisfies 2f-LE. In the second case, assume $S_j(\mathbf{r}) = \hat{\mathbf{k}}_j \cdot \mathbf{r}, |\hat{\mathbf{k}}_j| = 1$, with the additional conditions

$$\lim_{\theta\to 0} \hat{\mathbf{k}}_1 = \lim_{\theta\to 0} \hat{\mathbf{k}}_2 = \hat{\mathbf{k}}, \quad (9)$$

$$\frac{\hat{\mathbf{k}}_2 - \hat{\mathbf{k}}_1}{\theta\varepsilon} = \Delta\hat{\mathbf{k}}, \quad (10)$$

where $\Delta\hat{\mathbf{k}}$ is independent of θ, ε. Then the 2f-WD becomes

$$|A|^2 \left(\frac{\mathbf{x}}{k}\right) e^{i\Delta\hat{\mathbf{k}}\cdot\mathbf{x}} \delta(\mathbf{p} - \hat{\mathbf{k}}), \quad (11)$$

where β is absent due to the linear phase profile S_j.

Given, say, the Dirichlet boundary condition F imposed on the boundary $\partial\mathcal{D}$ of a phase-space domain $\mathcal{D}$, 2f-LE can be solved by the method of characteristics as shown below. The form of 2f-LE suggests the "gauge transformation" of 2f-WD

$$\mathfrak{W}^\varepsilon(\mathbf{x}, \mathbf{p}) = e^{-i\beta\mathbf{x}\cdot\mathbf{p}} W^\varepsilon(\mathbf{x}, \mathbf{p}) \quad (12)$$

which then satisfies the following more convenient equation:

$$\mathbf{p} \cdot \nabla_\mathbf{x} \mathfrak{W}^\varepsilon + i\beta|\mathbf{p}|^2 \mathfrak{W}^\varepsilon = -\frac{1}{2k\sqrt{\varepsilon}} (\nabla V)\left(\frac{\mathbf{x}}{\varepsilon k}\right) \cdot \nabla_\mathbf{p} \mathfrak{W}^\varepsilon \quad (13)$$

with the boundary condition that $\mathfrak{W}^\varepsilon(\mathbf{x}, \mathbf{p}) = \exp[-i\beta\mathbf{x}\cdot\mathbf{p}] F(\mathbf{x}, \mathbf{p}) \equiv \mathfrak{F}(\mathbf{x}, \mathbf{p})$ on $\partial\mathcal{D}$. In view of (12) $\mathfrak{W}^\varepsilon$ is the Fourier transform of the two-point function $U_1 \otimes U_2^*$ in the location difference (*i.e.* $\mathbf{y} + \beta\mathbf{x}$ measured in the unit of the central wavelength).

Consider the Hamiltonian system of time-reversed characteristic curves

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{x}^\varepsilon(t) = -\mathbf{p}^\varepsilon(t), \quad (14)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{p}^\varepsilon(t) = -\frac{1}{2k\sqrt{\varepsilon}} (\nabla V)\left(\frac{\mathbf{x}^\varepsilon(t)}{k\varepsilon}\right) \quad (15)$$

with $\mathbf{x}^\varepsilon(0) = \mathbf{x}, \mathbf{p}^\varepsilon(0) = \mathbf{p}$. Let $\tau^\varepsilon = \tau^\varepsilon(\mathbf{x}, \mathbf{p})$ be the first passage time when the trajectory $(\mathbf{x}^\varepsilon(\cdot), \mathbf{p}^\varepsilon(\cdot))$ hits the boundary of the phase-space domain $\mathcal{D}$. We then have

$$\mathfrak{W}^\varepsilon(\mathbf{x}, \mathbf{p}) = e^{-i\beta\int_0^{\tau^\varepsilon} |\mathbf{p}^\varepsilon(s)|^2 \mathrm{d}s - i\beta\mathbf{x}^\varepsilon(\tau^\varepsilon)\cdot\mathbf{p}^\varepsilon(\tau^\varepsilon)} F(\mathbf{x}^\varepsilon(\tau^\varepsilon), \mathbf{p}^\varepsilon(\tau^\varepsilon)).$$

Convergence to diffusion in momentum. – If V decorrelates sufficiently rapidly (see [16] for a precise formulation), the probability distribution of $(\mathbf{x}^\varepsilon(\cdot), \mathbf{p}^\varepsilon(\cdot))$ defined by (14), (15), converges weakly, as $\varepsilon \to 0$, to that of the Markov process $(\mathbf{x}(\cdot), \mathbf{p}(\cdot))$ where

$$\mathbf{x}(t) = \mathbf{x} - \int_0^t \mathbf{p}(s)\mathrm{d}s \quad (16)$$

and $\mathbf{p}(\cdot), \mathbf{p}(0) = \mathbf{p}$, is a diffusion process generated by the operator

$$\mathcal{A} = \frac{1}{4k} \nabla_\mathbf{p} \cdot \mathbf{D} \cdot \nabla_\mathbf{p}$$

with the (momentum) diffusion coefficient

$$\mathbf{D}(\mathbf{p}) = \pi \int \Phi(\mathbf{q}) \delta(\mathbf{p}\cdot\mathbf{q}) \mathbf{q} \otimes \mathbf{q} \mathrm{d}\mathbf{q}. \quad (17)$$

Writing $\mathbf{D}$ as

$$\mathbf{D}(\mathbf{p}) = \pi \int \Phi(\mathbf{q}) \delta(\mathbf{p}\cdot\mathbf{q}) \Pi(\mathbf{p})\mathbf{q} \otimes \Pi(\mathbf{p})\mathbf{q} \mathrm{d}\mathbf{q}, \quad (18)$$

where $\Pi(\mathbf{p})$ is the orthogonal projection onto the hyperplane perpendicular to $\mathbf{p}$ we see that the momentum diffusion process is concentrated on the sphere of radius $|\mathbf{p}|$. In other words, the limiting kinetic energy $|\mathbf{p}(t)|^2/2$ is preserved by the elastic scattering process. This observation will be useful for the subsequent calculation. To be consistent with the unity of the phase speed, we restrict $\mathbf{p}$ to the unit sphere $|\mathbf{p}| = 1$.

The consequence is the convergence of the ensemble average $\langle \mathfrak{W}^\varepsilon(\mathbf{x}, \mathbf{p}) \rangle$ to

$$\mathfrak{W}(\mathbf{x}, \mathbf{p}) \equiv \mathbb{E}_{\mathbf{x},\mathbf{p}} \left\{ e^{-i\beta|\mathbf{p}|^2\tau - i\beta\mathbf{x}(\tau)\cdot\mathbf{p}(\tau)} F(\mathbf{x}(\tau), \mathbf{p}(\tau)) \right\}, \quad (19)$$

where $\tau = \tau(\mathbf{x}, \mathbf{p})$ is the first passage time of the Markov process $(\mathbf{x}(t), \mathbf{p}(t))$ with $\mathbf{x}(0) = \mathbf{x}, \mathbf{p}(0) = \mathbf{p}$ and $\mathbb{E}_{\mathbf{x},\mathbf{p}}$ the corresponding average.

Now let $W(\mathbf{x}, \mathbf{p})$ be the solution of the following boundary value problem:

$$\mathbf{p} \cdot \nabla_\mathbf{x} W = \frac{1}{4k} (\nabla_\mathbf{p} - i\beta\mathbf{x}) \cdot \mathbf{D} \cdot (\nabla_\mathbf{p} - i\beta\mathbf{x}) W \quad (20)$$

with $W = F$ on $\partial\mathcal{D}$ and we will show that the solution of 2f-RT is the pointwise limit of the average 2f-WD. Equation (20) is our two-frequency radiative transfer (2f-RT) equation. Because we have considered the high-frequency asymptotics the scattering term takes the form of a second-order differential operator rather than the more familiar integral operator.

Let $\mathbf{p}(t)$ be the diffusion process generated by the generator $\mathcal{A}$ and define

$$\widetilde{W}(t, \mathbf{x}, \mathbf{p}) = \exp[-i\beta t|\mathbf{p}|^2 - i\beta\mathbf{x}\cdot\mathbf{p}] W(\mathbf{x}, \mathbf{p}). \quad (21)$$

By Dynkin's formula [17] we have that

$$\mathbb{E}_{\mathbf{x},\mathbf{p}} \left\{ \widetilde{W}(\tau, \mathbf{x}(\tau), \mathbf{p}(\tau)) \right\} = \widetilde{W}(0, \mathbf{x}, \mathbf{p}) + \mathbb{E}_{\mathbf{x},\mathbf{p}} \left\{ \int_0^\tau \left[\frac{\partial}{\partial s} - \mathbf{p}\cdot\nabla_\mathbf{x} + \mathcal{A}\right] \widetilde{W}(s, \mathbf{x}(s), \mathbf{p}(s)) \mathrm{d}s \right\}.$$

From (20), (21) it follows that

$$\left[\frac{\partial}{\partial t}-\mathbf{p}\cdot\nabla_{\mathbf{x}}+\mathcal{A}\right]\widetilde{W}=0$$

and

$$\mathfrak{W}(\mathbf{x},\mathbf{p})=\mathbb{E}_{\mathbf{x},\mathbf{p}}\left\{\widetilde{W}(\tau,\mathbf{x}(\tau),\mathbf{p}(\tau))\right\}=e^{-i\beta\mathbf{x}\cdot\mathbf{p}}W(\mathbf{x},\mathbf{p}). \tag{22}$$

Therefore, in view of (12), $W(\mathbf{x},\mathbf{p})$ is the pointwise limit of $\langle W^{\varepsilon}(\mathbf{x},\mathbf{p})\rangle$. It is straightforward to check that $\mathfrak{W}$ is the solution to the equation

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}\mathfrak{W}+i\beta|\mathbf{p}|^2\mathfrak{W}=\mathcal{A}\mathfrak{W}. \tag{23}$$

From (19) and (22) we obtain the probabilistic representation for W

$$W(\mathbf{x},\mathbf{p})=\mathbb{E}_{\mathbf{x},\mathbf{p}}\left\{e^{-i\beta\int_0^\tau\mathbf{x}(s)\mathrm{d}\mathbf{p}(s)}F(\mathbf{x}(\tau),\mathbf{p}(\tau))\right\}, \tag{24}$$

see ref. [17]. Expression (24) suggests a numerical solution procedure for 2f-RT by Monte Carlo simulation.

Isotropic medium. – Equation (23) clearly is translationally invariant in $\mathbf{x}$ due to the stationarity of the medium. If the medium is also statistically isotropic, then eq. (23) is rotationally invariant. To see this let us consider an isotropic spectral density $\Phi(\mathbf{p})=\Phi(|\mathbf{p}|)$. Then we have $\mathbf{D}=C|\mathbf{p}|^{-1}\Pi(\mathbf{p})$, where

$$C=\frac{\pi}{3}\int\delta\left(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\right)\Phi(|\mathbf{q}|)|\mathbf{q}|\mathrm{d}\mathbf{q} \tag{25}$$

is a constant. The coefficient C (and $\mathbf{D}$) has the dimension of inverse length while the variables $\mathbf{x}$ and $\mathbf{p}$ are dimensionless.

The resulting $\mathcal{A}$ is invariant with respect to rotation in $\mathbf{p}$. Hence if $\mathfrak{W}(\mathbf{x},\mathbf{p})$ is a solution to (23) then $\mathfrak{W}(R\mathbf{x},R\mathbf{p})$ is also a solution where R is any orthogonal matrix.

Spatial (frequency) spread and coherence bandwidth. – Through dimensional analysis, eq. (20) yields qualitative information about important physical parameters of the disordered medium. For this, let us assume for simplicity the isotropy of the medium as above.

Now consider the following change of variables:

$$\mathbf{x}=\sigma_x k\tilde{\mathbf{x}},\qquad \mathbf{p}=\sigma_p\tilde{\mathbf{p}}/k,\qquad \beta=\beta_c\tilde{\beta}, \tag{26}$$

where σ_x and σ_p are, respectively, the position spread and the spatial frequency spread, and β_c is the coherence bandwidth, also known as the Thouless frequency. Let us substitute (26) into eq. (20) and aim for the normalized form

$$\tilde{\mathbf{p}}\cdot\nabla_{\tilde{\mathbf{x}}}W=\left(\nabla_{\tilde{\mathbf{p}}}-i\tilde{\beta}\tilde{\mathbf{x}}\right)\cdot\frac{\Pi(\tilde{\mathbf{p}})}{|\tilde{\mathbf{p}}|}\left(\nabla_{\tilde{\mathbf{p}}}-i\tilde{\beta}\tilde{\mathbf{x}}\right)W. \tag{27}$$

The 1st term on the left side yields the first duality relation $\sigma_x/\sigma_p\sim 1/k^2$. The balance of the terms in each pair of the parentheses yields the second duality relation $\sigma_x\sigma_p\sim 1/\beta_c$ whose left-hand side is the *space-spread-bandwidth product*. Finally, the removal of the constant C determines σ_p from which σ_x and β_c can be determined by using the duality relations. We obtain

$$\sigma_p\sim k^{2/3}C^{1/3},\quad \sigma_x\sim k^{-4/3}C^{1/3},\quad \beta_c\sim k^{2/3}C^{-2/3}. \tag{28}$$

Spatially anisotropic media. – Forward-scattering approximation, also called paraxial approximation, is valid when back-scattering is negligible and, as we show below, this is the case for spatially anisotropic media fluctuating slowly in the (longitudinal) direction of propagation.

Let z denote the longitudinal coordinate and $\mathbf{x}_\perp$ the transverse coordinates. Let p and $\mathbf{p}_\perp$ denote the longitudinal and transverse components of $\mathbf{p}\in\mathbb{R}^3$, respectively. Let $\mathbf{q}=(q,\mathbf{q}_\perp)\in\mathbb{R}^3$ be likewise defined. Consider now a highly anisotropic spectral density for a medium fluctuating much more slowly in the longitudinal direction, *i.e.* replacing $\Phi(\mathbf{q})$ in (17) by $\eta^{-1}\Phi\left(\eta^{-1}q,\mathbf{q}_\perp\right)$, $\eta\ll 1$, which, in the limit $\eta\to 0$, tends to

$$\delta(q)\int\mathrm{d}w\Phi\left(w,\mathbf{q}_\perp\right). \tag{29}$$

We then obtain the transverse diffusion coefficient

$$\mathbf{D}_\perp(\mathbf{p}_\perp)=\pi\int\mathrm{d}\mathbf{q}_\perp\int\mathrm{d}w\Phi(w,\mathbf{q}_\perp)\delta(\mathbf{p}_\perp\cdot\mathbf{q}_\perp)\mathbf{q}_\perp\otimes\mathbf{q}_\perp,$$

whereas the longitudinal diffusion coefficient now vanishes. In other words, the longitudinal momentum is decoupled from the transverse momentum and is not directly affected by the multiple scattering process.

For simplicity we assume the transverse isotropy, *i.e.* $\Phi(w,\mathbf{p}_\perp)=\Phi(w,|\mathbf{p}_\perp|)$, so that $\mathbf{D}_\perp=C_\perp|\mathbf{p}_\perp|^{-1}\Pi_\perp(\mathbf{p}_\perp)$, where

$$C_\perp=\frac{\pi}{2}\int\delta\left(\frac{\mathbf{p}_\perp}{|\mathbf{p}_\perp|}\cdot\frac{\mathbf{q}_\perp}{|\mathbf{q}_\perp|}\right)\Phi(w,|\mathbf{q}_\perp|)|\mathbf{q}_\perp|\mathrm{d}w\mathrm{d}\mathbf{q}_\perp$$

is a constant and $\Pi_\perp(\mathbf{p}_\perp)$ is the orthogonal projection onto the line perpendicular to $\mathbf{p}_\perp$. Hence eq. (20) reduces to

$$[p\partial_z+\mathbf{p}_\perp\cdot\nabla_{\mathbf{x}_\perp}]\bar{W}=$$
$$\frac{C_\perp}{4k}\left(\nabla_{\mathbf{p}_\perp}-i\beta\mathbf{x}_\perp\right)\cdot\frac{\Pi_\perp(\mathbf{p}_\perp)}{|\mathbf{p}_\perp|}\left(\nabla_{\mathbf{p}_\perp}-i\beta\mathbf{x}_\perp\right)\bar{W}. \tag{30}$$

Note that the longitudinal momentum p plays the role of a parameter in eq. (30) which then can be solved in the direction of increasing z as an evolution equation with initial data given at a fixed z.

As before we can obtain the scaling behaviors of spatial spread, coherence length and coherence bandwidth by dimensional analysis. Let σ_* be the spatial spread in the transverse coordinates $\mathbf{x}_\perp$, ℓ_c the coherence length in the transverse dimensions, β_c the coherence bandwidth and L the distance of propagation. We then seek the following change of variables:

$$\tilde{\mathbf{x}}_\perp = \frac{\mathbf{x}_\perp}{\sigma_* k}, \quad \tilde{\mathbf{p}}_\perp = \mathbf{p}_\perp k\ell_c, \quad \tilde{z} = \frac{z}{Lk}, \quad \tilde{\beta} = \frac{\beta}{\beta_c} \tag{31}$$

to remove all the physical parameters from (30). Following the same line of reasoning, we obtain that $\ell_c\sigma_* \sim L/k$, $\sigma_*/\ell_c \sim 1/\beta_c$, $\ell_c \sim C_\perp^{-1/3}L^{-1/3}k^{-1}$ and hence $\sigma_* \sim C_\perp^{1/3}L^{4/3}$, $\beta_c \sim C_\perp^{-2/3}L^{-5/3}k^{-1}$.

Small-scale asymptotic. – On the scale below the transport mean-free-path ℓ_* the scattering is extremely anisotropic and the scattering amplitude is highly peaked in the forward direction. This observation leads to a paraxial approximation of 2f-RT which turns out to be analytically solvable.

Let z be the direction of propagation of a collimated beam. On the scale below ℓ_* the 2f-WD near the source point would be highly concentrated at the longitudinal momentum, say, $p=1$. Hence we may assume that the projection $\Pi(\mathbf{p})$ in (18) is effectively just the projection onto the transverse plane coordinated by $\mathbf{x}_\perp$ and we can approximate eq. (20) by

$$[\partial_z + \mathbf{p}_\perp \cdot \nabla_{\mathbf{x}_\perp}]W = \frac{C_\perp}{4k}\left(\nabla_{\mathbf{p}_\perp} - i\beta\mathbf{x}_\perp\right)^2 W, \tag{32}$$

where

$$C_\perp = \frac{\pi}{2}\int \Phi(0,\mathbf{q}_\perp)|\mathbf{q}_\perp|^2 \mathrm{d}\mathbf{q}_\perp.$$

Equation (32) is another form of paraxial approximation for which only the one-sided (incoming) boundary condition (at $z=\text{const}$) is needed.

We use the change of variables (31) with $L=\ell_*$ and obtain $\ell_c \sim k^{-1}\ell_*^{-1/2}C_\perp^{-1/2}$, $\sigma_* \sim \ell_*^{3/2}C_\perp^{1/2}$, $\beta_c \sim k^{-1}C_\perp^{-1}\ell_*^{-2}$ from eq. (32). The transport mean-free-path ℓ_* can be determined by setting $\ell_c \sim 1$, *i.e.* $\ell_* \sim k^{-2}C_\perp^{-1}$. Performing the inverse Fourier transform in $\tilde{\mathbf{p}}$ on the rescaled equation we obtain

$$\partial_{\tilde{z}}\Gamma - i\nabla_{\tilde{\mathbf{y}}_\perp}\cdot\nabla_{\tilde{\mathbf{x}}_\perp}\Gamma = -\left|\tilde{\mathbf{y}}_\perp + \tilde{\beta}\tilde{\mathbf{x}}_\perp\right|^2\Gamma, \tag{33}$$

which is the governing equation for the two-frequency coherence Γ. By a simple change of coordinates, eq. (33) can be converted into a form similar to the time-dependent Schrödinger equation with a (purely imaginary) quadratic potential and then solved analytically. Let $\Delta\mathbf{r} = \mathbf{y}_\perp + \tilde{\beta}\mathbf{x}_\perp$ and $\Delta\mathbf{r}' = \mathbf{y}'_\perp + \tilde{\beta}\mathbf{x}'_\perp$ be the field point offset and the source point offset, respectively, measured in the unit of central wavelength. The propagator for the initial value problem from the source point $(\tilde{\mathbf{x}}_\perp, \Delta\mathbf{r})$ to the field point

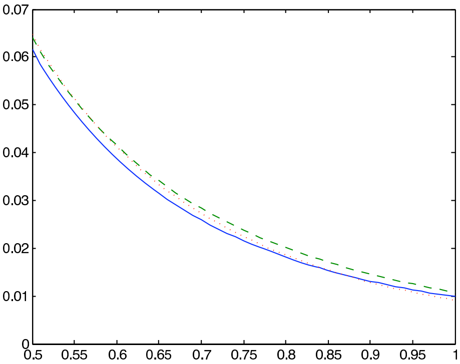

Fig. 1: The absolute value of (34) as a function of $\tilde{z} \in [0.5, 1]$ for $\Delta\mathbf{r} = \Delta\mathbf{r}' = 1, \tilde{\beta} = 0.3, 1, 3.3$ in solid, dashed and dotted lines, respectively.

$(\mathbf{x}'_\perp, \Delta\mathbf{r}')$ is given by [14]

$$\begin{aligned} &\frac{(i4\tilde{\beta})^{1/2}}{(2\pi)^2\tilde{z}\sinh\left[(i4\tilde{\beta})^{1/2}\tilde{z}\right]} e^{\frac{1}{i4\tilde{\beta}\tilde{z}}\left|\Delta\mathbf{r} - 2\tilde{\beta}\tilde{\mathbf{x}}_\perp - \Delta\mathbf{r}' + 2\tilde{\beta}\mathbf{x}'_\perp\right|^2} \\ &\times e^{-\frac{\coth\left[(i4\tilde{\beta})^{1/2}\tilde{z}\right]}{(i4\tilde{\beta})^{1/2}}\left|\Delta\mathbf{r} - \frac{\Delta\mathbf{r}'}{\cosh\left[(i4\tilde{\beta})^{1/2}\tilde{z}\right]}\right|^2} \\ &\times e^{-\frac{\tanh\left[(i4\tilde{\beta})^{1/2}\tilde{z}\right]}{(i4\tilde{\beta})^{1/2}}\left|\Delta\mathbf{r}'\right|^2} \end{aligned} \tag{34}$$

which converges, in the limit $\tilde{\beta}\downarrow 0$, to the propagator for $\tilde{\beta}=0$

$$(2\pi\tilde{z})^{-2} e^{\frac{i}{\tilde{z}}(\tilde{\mathbf{x}}_\perp - \mathbf{x}'_\perp)\cdot(\Delta\mathbf{r} - \Delta\mathbf{r}')} e^{-\frac{\tilde{z}}{3}(|\Delta\mathbf{r}|^2 + \Delta\mathbf{r}\cdot\Delta\mathbf{r}' + |\Delta\mathbf{r}'|^2)}. \tag{35}$$

The quadratic-in-$\Delta\mathbf{r}$ nature of the exponents appearing in (34)-(35) is the consequence of the paraxial approximation. Expression (35) is related to the asymptotic solution of the Schwarzschild-Milne equation in the case of very anisotropic scattering [18].

In view of (9)-(11), to get the correlation of two incident plane waves we simply express (34) in the variables $\tilde{\mathbf{x}}_\perp, \mathbf{x}'_\perp$ and $\tilde{\mathbf{y}}_\perp, \mathbf{y}'_\perp$ and integrate it with $e^{i\Delta\hat{\mathbf{k}}\cdot\mathbf{x}'_\perp}e^{i\hat{\mathbf{k}}\cdot\mathbf{y}'_\perp}$.

The functional form of (34) in its dependence on $\tilde{\beta}$ and $\tilde{z}$ is the main characteristic of the sub-ℓ_*-scale behavior (see fig. 1).

Conclusion and discussion. – The main contribution of the present letter is the rigorous derivation of the 2f-RT equation (20) governing 2f-WD in disordered media and the probabilistic representation (24). As a result, by (2) we can express the two-space-time correlation as

$$\begin{aligned}\langle u(t_1,\mathbf{x}_1)u^*(t_2,\mathbf{x}_2)\rangle \sim \int_{|\mathbf{p}|=1} & e^{i\mathbf{p}\cdot(\mathbf{y}+\beta\mathbf{x})}e^{ik(t_2-t_1)/(\varepsilon\theta)} \\ & \times e^{ik\beta(t_1+t_2)/2}\mathfrak{W}(\mathbf{x},\mathbf{p})\mathrm{d}k\mathrm{d}\beta\mathrm{d}\Omega(\mathbf{p})\end{aligned}$$

with $\mathbf{x} = k(\mathbf{x}_1+\mathbf{x}_2)/2$, $\mathbf{y} = k(\mathbf{x}_1 - \mathbf{x}_2)/(\theta\varepsilon)$ where $\mathfrak{W}$ is the solution to eq. (23). The medium characteristic enters the Fokker-Planck–like equation (20) only through the momentum diffusion coefficient (17). By dimensional analysis with (20) and its variants we obtain the scaling behavior of spatial spread, coherence length and coherence bandwidth for isotropic and anisotropic media. We also show that the paraxial regime is valid for anisotropic scattering, giving rise to two forms of paraxial 2f-RT equations. Finally, by solving one of the paraxial equation (32) we obtain the precise profile of the space-frequency correlation on the scale below the transport mean-free-path.

Let us compare our results, especially (34), with the existing results in the literature which mostly concern with the bulk behavior of the space-frequency correlations.

Since the bulk behavior concerns the scales larger than the transport mean-free-path the existing results are mostly based on the diffusion approximation to the displacement process $\mathbf{x}(t)$ or the random-matrix method (see, *e.g.*, [4,19] and references therein). The diffusion regime represents an isotropic scattering under the condition of equipartition of energy while the small-scale asymptotic (34) describes an extremely anisotropic scattering.

Clearly, the diffusion approximation is unsuitable for evaluating (24) because of the presence of the integral with respect to the momentum process $\mathbf{p}(t)$. Therefore to get the two-frequency coherence, the notion of the interference of diffusions is invoked via diagrammatic techniques, see the review [4].

In the diffusion approximation for isotropic media, the (dimentionless) $\mathbf{x}$-diffusion coefficient D_* can be derived from (20) with $\beta = 0$

$$D_* = \frac{4k|\mathbf{p}|^5}{3C}. \tag{36}$$

With (36) and (28) we can rewrite the scaling behaviors of the spatial spread, the spatial frequency spread and the coherence bandwidth as $\sigma_x \sim k^{-1}D_*^{-1/3}$, $\sigma_p \sim kD_*^{-1/3}$, $\beta_c \sim D_*^{2/3}$.

The short-range correlation C_1 of wave intensities propagating through disordered media is manifest in the speckle pattern. C_1 can be obtained by squaring the two-frequency coherence of the wave fields [20] and the commonly accepted form is $\exp[-2\sqrt{2\tilde{\beta}}]$ which is just the large $\tilde{\beta}$ asymptotic of the squared factor $|\sinh[(i4\tilde{\beta})^{1/2}\tilde{z}]|^{-2}$ at $\tilde{z} = 1$ (see, *e.g.*, [21–23]).

More precisely, the squared absolute value of (34) for $\tilde{z} = 1$ and median to large $\tilde{\beta}$ is approximately given by

$$\frac{4\tilde{\beta}}{(2\pi)^4} e^{-2\sqrt{2\tilde{\beta}}} e^{-\frac{|\Delta\mathbf{r}|^2}{\sqrt{2\tilde{\beta}}}} e^{-\frac{|\Delta\mathbf{r}'|^2}{\sqrt{2\tilde{\beta}}}}. \tag{37}$$

Expression (37) is essentially the same as the paraxial approximation of the short-range correlation C_1 reviewed in [4]. The multiplicative nature of (37)'s functional form in $\Delta\mathbf{r}$ and $\Delta\mathbf{r}'$ is consistent with the same structure in the short-range intensity correlation $C_1 = A(\Delta k)F(\Delta\mathbf{r})F(\Delta\mathbf{r}')$ discovered in [24]. Again, the Gaussian form in (37) is different from the form factor F in [24] due to the paraxial approximation made in obtaining (37).

The long- and infinite-range correlations, represented by C_2 and C_3 respectively, can also be obtained by our method [4,24–27]. The calculation is much more involved and will be presented elsewhere.

REFERENCES

[1] Mandel L. and Wolf E., *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge) 1995.

[2] Ishimaru A., *Wave Propagation and Scattering in Random Media,* Vols. **I** and **II** (Academic, New York) 1978.

[3] Berkovits R. and Feng A., *Phys. Rep.*, **238** (1994) 135.

[4] van Rossum M. C. W. and Nieuwenhuizen Th. M., *Rev. Mod. Phys.*, **71** (1999) 313.

[5] Samelsohn G. and Freilikher V., *Phys. Rev. E*, **65** (2002) 046617.

[6] Bronshtein A., Lu I. T. and Mazar R., *Phys. Rev. E*, **69** (2004) 016607.

[7] Chandrasekhar S., *Radiative Transfer* (Dover Publications, New York) 1960.

[8] Wigner E., *Phys. Rev.*, **40** (1932) 749.

[9] Dragoman D., *Prog. Opt.*, **37** (1997) 1.

[10] Forbes G. W., Man'ko V. I., Ozaktas H. M., Simon R. and Wolf K. B., *J. Opt. Soc. Am. A*, **17** (2000) 2274.

[11] Schleich W. P., *Quantum Optics in Phase Space* (Wiley-VCH, Berlin, New York) 2001.

[12] Kravtsov Yu. A. and Apresyan L. A., *Prog. Opt.*, **36** (1996) 179.

[13] Fannjiang A. C., *C. R. Math.*, **342** (2006) 109.

[14] Fannjiang A. C., *J. Opt. Soc. Am. A*, **24** (2007) 2248.

[15] Spohn H., *Rev. Mod. Phys.*, **53** (1980) 569.

[16] Kesten H. and Papanicolaou G. C., *Commun. Math. Phys.*, **78** (1980) 19.

[17] Stroock D. W. and Varadhan S. R. S., *Multidimensional Diffusion Processes* (Springer-Verlag, Berlin) 1979.

[18] Amic E., Luck J. M. and Nieuwenhuizen Th. M., *J. Phys. A: Math. Gen.*, **29** (1996) 4915.

[19] Froute-Pérez L. S., Yépez M., Mello P. A. and Sáenz J. J., *Phys. Rev. E*, **75** (2007) 031113.

[20] Pnini R. and Shapiro B., *Phys. Rev. B*, **39** (1989) 6986.

[21] Shapiro B., *Phys. Rev. Lett.*, **57** (1986) 2168.

[22] Feng S., Kane C., Lee P. A. and Stone A. D., *Phys. Rev. Lett.*, **61** (1988) 834.

[23] Genack A. Z., *Phys. Rev. Lett.*, **58** (1987) 2043.

[24] Sebbah P., Hu P. B., Genack A. Z., Pnini R. and Shapiro B., *Phys. Rev. Lett.*, **88** (2002) 123901.

[25] Shapiro B., *Phys. Rev. Lett.*, **83** (1999) 4733.

[26] Skipetrov S. E. and Maynard R., *Phys. Rev. B*, **62** (2000) 886.

[27] García-Martín A., Scheffold F., Nieto-Vesperinas M. and Sáenz J. J., *Phys. Rev. Lett.*, **88** (2002) 143901.