Math 108, Fall 2013.
Dec. 10, 2013.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

Instructions: Each of the 8 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK AND FULLY JUSTIFY YOUR ANSWERS TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 10 pages (including this one) with 8 problems.

1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

1. For each statement below, determine whether it is true or false for the universe $\mathbb{N} = \{1, 2, 3, \ldots\}$.
(a) $(\forall x)(\exists y)(y > x)$

True. Take $y = x+1$.

(b) $(\forall x)(\exists y)((y \leq 50) \wedge (y > x))$

False. Counterexample is $x = 50$. There is no y such that $y \leq 50$ and $y > x = 50$.

(c) $(\forall x)(\forall y)(((y < 5) \wedge (x < 5)) \implies (x + y \leq 8))$

True. If $y < 5$ and $x < 5$, then $y \leq 4$ and $x \leq 4$ and so $x+y \leq 8$.

(d) $(\exists x)(\forall y)((y \geq x) \implies y^2 \geq 9y))$

True. Take $x = 9$. Then $y \geq x = 9$ implies $y^2 = y \cdot y \geq 9y$.

(e) $(\exists x)(\exists y)(\forall z)((z < x + y) \vee (z > 3x + 4y))$

False. Negation $(\forall x)(\forall y)(\exists z)((z \geq x+y) \wedge (z \leq 3x+4y))$ is true, as we can take $z = x+y$.
Then $z \geq x+y$ and, as $x+y \leq 3x+4y$, $z \leq 3x+4y$.

2. Assume that A, B, C are arbitrary subsets of $\mathbb{N}$. For each statement below, prove it or provide a counterexample.

(a) Prove: If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

Take $x \in A \cup B$. Then either:

- $x \in A \subseteq C$, so $x \in C$; or
- $x \in B \subseteq C$, so $x \in C$.

In either case, $x \in C$.

(b) Prove: If $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$.

Take $x \in A$. We need to show that $x \in C$.
Assume otherwise, $x \notin C$. Then, as $x \in A \subseteq B \cup C$, it follows that $x \in B$. But this means $x \in A \cap B \neq \emptyset$. This contradiction implies $x \in C$, and so $A \subseteq C$.

(c) Give a counterexample: If $C \subseteq A \cup B$, then $A \cap B \subseteq C$

Take $A = B = \{1\}$, $C = \emptyset$. Then $\emptyset \subseteq A \cup B$ (no matter what A and B are) and $A \cap B = \{1\} \not\subseteq C$.

Problem 2, continued.

(d) Give a counterexample: $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$

$A = \{1\}$, $B = \{2\}$. Then $\{1,2\} = A \cup B \in \mathcal{P}(A \cup B)$, but $\{1,2\} \not\subseteq \{1\}$ and $\{1,2\} \not\subseteq \{2\}$, so $\{1,2\} \notin \mathcal{P}(A)$ and $\{1,2\} \notin \mathcal{P}(B)$, and then $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

(e) Prove: If A is denumerable and B is finite, then $A - B$ is denumerable.

$A - B$ is <u>countable</u>, as a subset of countable set A, i.e., $A - B \subseteq A$.

$A - B$ is <u>infinite</u>; if it were finite, then $A = B \cup (A - B)$ would be a union of two finite sets, thus finite.

Therefore, $A - B$ is denumerable, as infinite countable set.

(f) Give a counterexample: If A is denumerable and B is denumerable, then $A \cap B$ is denumerable.

Take $A = \{2, 4, 6, 8, \ldots\} = \{2k : k \in \mathbb{N}\}$

$B = \{1, 3, 5, \ldots\} = \{2k-1, k \in \mathbb{N}\}$

Then A and B are denumerable, but $A \cap B = \emptyset$, thus finite.

3. Prove (by induction or any other correct method you know) that for every $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} 5^i = \frac{1}{4}5^{n+1} - \frac{1}{4}.$$

$(n=1)$ $\quad 5^0 + 5^1 = \frac{1}{4}5^2 - 1$, Both sides equal 6 ,

$(n \to n+1)$ $\quad \sum_{i=0}^{n+1} 5^i = \sum_{i=0}^{n} 5^i + 5^{n+1}$

$$\underset{(IH)}{=} \frac{1}{4}5^{n+1} - \frac{1}{4} + 5^{n+1}$$

$$= \left(\frac{1}{4} + 1\right) 5^{n+1} - \frac{1}{4}$$

$$= \frac{5}{4} \cdot 5^{n+1} - \frac{1}{4} = \underline{\underline{\frac{1}{4}5^{n+2} - \frac{1}{4}}} \; . \quad \square$$

4. Prove that $n^2 + 5n + 4$ is even for every $n \in \mathbb{Z}$. (*Hint*: consider two cases.)

Case 1: n even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Then

$$n^2+5n+4 = (2k)^2 + 5 \cdot 2k + 4$$
$$= 4k^2 + 10k + 4$$
$$= 2(2k^2+5k+2)$$

is an even number.

Case 2: n odd. Then $n = 2k-1$, for some $k \in \mathbb{Z}$. Then

$$n^2+5n+4 = (2k-1)^2 + 5(2k-1) + 4$$
$$= 4k^2 - 4k \cancel{+1} + 10k \cancel{-5} \cancel{+4}$$
$$= 4k^2 - 6k$$
$$= 2(2k^2-3k)$$

is an even number. $\square$

5. Prove the following two statements. You may use (a) to prove (b).
(a) The number $\sqrt{3}$ is irrational.

Assume $\sqrt{3} \in \mathbb{Q}$; then $\sqrt{3} = \frac{p}{q}$, where p and q are natural numbers without a common factor, then $3 = \frac{p^2}{q^2}$, $p^2 = 3q^2$ and $3 \mid p^2$. Thus $3 \mid p$ and $p = 3k$ for some k, but then $9k^2 = 3q^2$, $q^2 = 3k^2$, $3 \mid q^2$ and $3 \mid q$. So p and q have a common factor 3, a contradiction.

(b) The number $\frac{1}{2}\sqrt{3} + 5$ is irrational.

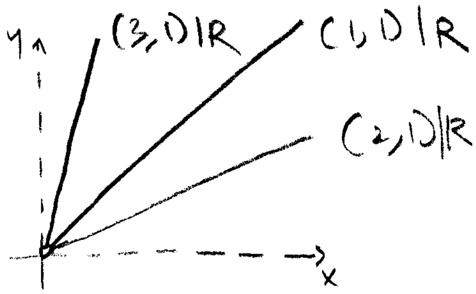
Assume $\frac{1}{2}\sqrt{3} + 5 \in \mathbb{Q}$. Then $\frac{1}{2}\sqrt{3} + 5 = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$, and then

$$\sqrt{3} = \left(\frac{p}{q} - 5\right) \cdot 2 = \frac{2p - 10q}{q}$$

As $2p - 10q \in \mathbb{Z}$ and $q \in \mathbb{Z}$, $\sqrt{3} \in \mathbb{Q}$, a contradiction.

6. Let A be the open first quadrant in the plane, that is, $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : (x > 0) \wedge (y > 0)\}$. Define the relation R on A as follows: $(x, y)R(a, b)$ if and only if $xb = ya$. Prove that this is an equivalence relation and describe the resulting partition. In particular, sketch the equivalence classes of $(1, 1)$, $(2, 1)$, and $(1, 3)$. (To show that R is an equivalence relation, you may use any method we covered in the lecture.)

Let $f : A \to \mathbb{R}$ be given by $f(x,y) = \frac{x}{y}$. Then $(x,y) R (a,b)$ if and only if $f(x,y) = f(a,b)$, so R is an equivalence relation by a thm. from class.



The partition divides A into lines through the (open) half-lines, to be more precise).

$(1,1)|R = \{(x,y) \in A : x = y\}$
$(2,1)|R = \{(x,y) \in A : x = 2y\}$
$(1,3)|R = \{(x,y) \in A : 3x = y\}$

7. Assume A, B, and C are arbitrary sets and $f : A \to B$ and $g : B \to C$ are arbitrary functions. Assume also that D and E are arbitrary subset of A. For each statement below, prove it or provide a counterexample.

(a) Prove: If $f(D) \cap f(E) = \emptyset$ then $D \cap E = \emptyset$.

Contrapositive:
Assume $D \cap E \neq \emptyset$. Then $\exists x \in D \cap E$, but then $f(x) \in f(D)$ and $f(x) \in f(E)$, and so $f(D) \cap f(E) \neq \emptyset$.

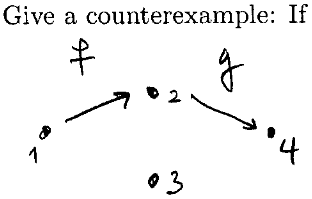
(b) Prove: If f is one-to-one and $D \cap E = \emptyset$, then $f(D) \cap f(E) = \emptyset$.

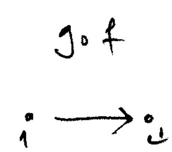
Assume f is one-to-one, and that $f(D) \cap f(E) \neq \emptyset$. Then there $\exists y \in f(D) \cap f(E)$. Then $y = f(x_1)$ for some $x_1 \in D$ and $y = f(x_2)$ for some $x_2 \in E$. But f is one-to-one, so $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Thus $x_1 \in D$ and $x_1 = x_2 \in E$, $x_1 \in D \cap E$. So $D \cap E \neq \emptyset$.

(c) Prove: If f and g are both onto, then $g \circ f$ is onto.

Take a $z \in C$. Then there exists a $y \in B$ so that $g(y) = z$ and then an $x \in A$ so that $f(x) = y$. Thus $g(f(x)) = z$, i.e. $(g \circ f)(x) = z$.

(d) Give a counterexample: If $g \circ f$ is onto, then f is onto.





f is not onto

$g \circ f$ is onto

8. Determine the cardinality of each of the following sets. Give the result as either 0, a natural number, $\aleph_0$, or **c**.
(a) $\{1,2,3,4\}^{\{1,2\}}$

$$4^2 = 16$$

(b) $\{1,2,3\} \times \{1,2\}$

$$3 \cdot 2 = 6$$

(c) $\mathcal{P}(\{1,2,3\} \times \{1,2\})$

$$2^6 = 64$$

(d) $\{\frac{7}{2^n} : n \in \mathbb{N}\}$ $\aleph_0$, as this is an infinite subset of $\mathbb{Q}$ (which is denumerable).

(e) $[2,3]$ c as $(0,1) \approx (2,3) \subseteq [2,3] \Rightarrow \overline{\overline{[2,3]}} \geq c$
and $[2,3] \subseteq \mathbb{R} \approx (0,1) \Rightarrow \overline{\overline{[2,3]}} \leq c$
Cantor-Bernstein Thm. implies $\overline{\overline{[2,3]}} = c$.

(f) $\mathbb{Q} \times \mathbb{N}$ $\approx \mathbb{N} \times \mathbb{N} \approx \mathbb{N}$

$\aleph_0$

(g) $\mathbb{R} \times \mathbb{N}$ c

$\mathbb{R} \times \mathbb{N} \subseteq \mathbb{R} \times \mathbb{R} \approx \mathbb{R}$, thus $\overline{\overline{\mathbb{R} \times \mathbb{N}}} \leq c$.
$\mathbb{R} \approx \mathbb{R} \times \{1\} \subseteq \mathbb{R} \times \mathbb{N}$, thus $\overline{\overline{\mathbb{R} \times \mathbb{N}}} \geq c$.
By Cantor-Bernstein, $\overline{\overline{\mathbb{R} \times \mathbb{N}}} = c$.