Thus, $\sim P \vee Q$ is an abbreviation for $(\sim P) \vee Q$, but $\sim(P \vee Q)$ is the only way to write the negation of $P \vee Q$. Here are some other examples:

$P \vee Q \wedge R$ abbreviates $P \vee (Q \wedge R)$.
$P \wedge \sim Q \vee \sim R$ abbreviates $[P \wedge (\sim Q)] \vee (\sim R)$.
$\sim P \vee \sim Q$ abbreviates $(\sim P) \vee (\sim Q)$.
$\sim P \wedge \sim R \vee \sim P \wedge R$ abbreviates $[(\sim P) \wedge (\sim R)] \vee [(\sim P) \wedge R]$.

When the same connective is used several times in succession, parentheses may be omitted. We reinsert parentheses from the left, so that $P \vee Q \vee R$ is really $(P \vee Q) \vee R$. For example, $R \wedge P \wedge \sim P \wedge Q$ abbreviates $[(R \wedge P) \wedge (\sim P)] \wedge Q$, whereas $R \vee P \wedge \sim P \vee Q$, which does not use the same connective consecutively, abbreviates $(R \vee [P \wedge (\sim P)]) \vee Q$. Leaving out parentheses is not required; some propositional forms are much easier to read with a few well-chosen "unnecessary" parentheses.

Exercises 1.1

1. Use your knowledge of number systems to determine whether each is true or false.

(a) 11 is a rational number.
★ **(b)** 5π is a rational number.
(c) There are exactly 3 prime numbers between 40 and 50.
(d) There are exactly 5 prime numbers less than 10.
(e) 29 is a composite number.
(f) 0 is a natural number.
★ **(g)** $(5 + 2i)(5 - 2i)$ is a real number.
(h) 18 is a multiple of 12.

2. Which of the following are propositions? Give the truth value of each proposition.

(a) What time is dinner?
(b) It is not the case that $5 + \pi$ is not a rational number.
★ **(c)** $x/2$ is a rational number.
(d) $2x + 3y$ is a real number.
(e) Either $3 + \pi$ is rational or $3 - \pi$ is rational.
★ **(f)** Either 2 is rational and π is irrational, or 2π is rational.
(g) Either 5π is rational and 4.9 is rational, or 3π is rational.
(h) $-\frac{1}{2}$ is rational, and either $3\pi < 10$ or $3\pi > 15$.
(i) It is not the case that 39 is prime, or that 64 is a power of 2.
(j) There are more than three false statements in this book and this statement is one of them.

3. Make truth tables for each of the following propositional forms.

★ **(a)** $P \wedge \sim P$.
(b) $P \vee \sim P$.
★ **(c)** $P \wedge \sim Q$.
(d) $P \wedge (Q \vee \sim Q)$.
★ **(e)** $(P \wedge Q) \vee \sim Q$.
(f) $\sim(P \wedge Q)$.
(g) $(P \vee \sim Q) \wedge R$.
(h) $\sim P \wedge \sim Q$.

★ **(i)** $P \wedge (Q \vee R)$.
(j) $(P \wedge Q) \vee (P \wedge R)$.
(k) $P \wedge P$.
(l) $(P \wedge Q) \vee (R \wedge \sim S)$.

4. If P, Q, and R are true while S and T are false, which of the following are true?
★ **(a)** $Q \wedge (R \wedge S)$.
(b) $Q \vee (R \wedge S)$.
★ **(c)** $(P \vee Q) \wedge (R \vee S)$.
(d) $(\sim P \vee \sim Q) \vee (\sim R \vee \sim S)$.
(e) $\sim P \vee \sim Q$.
★ **(f)** $(\sim Q \vee S) \wedge (Q \vee S)$.
★ **(g)** $(P \vee S) \wedge (P \vee T)$.

5. Use truth tables to prove the remaining parts of Theorem 1.1.1.

6. Which of the following pairs of propositional forms are equivalent?
★ **(a)** $P \wedge P$, P.
(b) $P \vee P$, P.
★ **(c)** $P \wedge Q$, $Q \wedge P$.
(d) $(\sim P) \vee (\sim Q)$, $\sim (P \vee \sim Q)$.
★ **(e)** $\sim P \wedge \sim Q$, $\sim (P \wedge \sim Q)$.
(f) $\sim (P \wedge Q)$, $\sim P \wedge \sim Q$.
★ **(g)** $(P \wedge Q) \vee R$, $P \wedge (Q \vee R)$.
(h) $(P \wedge Q) \vee R$, $P \vee (Q \wedge R)$.

7. Determine the propositional form and truth value for each of the following:
(a) It is not the case that 2 is odd.
(b) $f(x) = e^x$ is increasing and concave up.
(c) Both 7 and 5 are factors of 70.
(d) Perth or Panama City or Pisa is located in Europe.

8. P, Q, and R are propositional forms, and P is equivalent to Q, and Q is equivalent to R. Prove that
★ **(a)** Q is equivalent to P.
(b) P is equivalent to R.
(c) $\sim Q$ is equivalent to $\sim P$.

9. Use a truth table to determine whether each of the following is a tautology, a contradiction, or neither.
(a) $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.
(b) $\sim (P \wedge \sim P)$.
★ **(c)** $(P \wedge Q) \vee (\sim P \vee \sim Q)$.
(d) $(A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B) \vee (\sim A \wedge \sim B)$.
(e) $(Q \wedge \sim P) \wedge \sim (P \wedge R)$.
(f) $P \vee [(\sim Q \wedge P) \wedge (R \vee Q)]$.

10. Suppose A is a tautology and B is a contradiction. Are the following tautologies, contradictions, or neither?
★ **(a)** $A \wedge B$.
(b) $A \wedge \sim B$.
★ **(c)** $A \vee B$.
(d) $\sim (\sim A \wedge B)$.

11. Give a useful denial of each statement.
★ **(a)** x is a positive integer. (Assume that x is some fixed integer.)
(b) Cleveland will win the first game or the second game.
★ **(c)** $5 \geq 3$.
(d) 641,371 is a composite integer.
★ **(e)** Roses are red and violets are blue.
(f) T is not bounded or T is compact. (Assume that T is a fixed object.)
(g) M is odd and one-to-one. (Assume that M is some fixed function.)

(h) The function f has positive first and second derivatives at x_0. (Assume that f is a fixed function and x_0 is a fixed real number.)
(i) The function g has a relative maximum at $x = 2$ or $x = 4$ and a relative minimum at $x = 3$. (Assume that g is a fixed function.)
(j) Neither $z < s$ nor $z \leq t$ is true. (Assume that z, s, and t are fixed real numbers.)
(k) R is transitive but not reflexive. (Assume that R is a fixed object.)

12. Restore parentheses to these abbreviated propositional forms.
(a) $\sim\sim P \vee \sim Q \wedge \sim S$.
(b) $Q \wedge \sim S \vee \sim(\sim P \wedge Q)$.
(c) $P \wedge \sim Q \ \vee \sim P \wedge \sim R \vee \sim P \wedge S$.
(d) $\sim P \vee Q \wedge \sim\sim P \wedge Q \vee R$.

13. Other logical connectives between two propositions P and Q are possible.
(a) The word *or* is used in two different ways in English. We have presented the truth table for $\vee$, the **inclusive or,** whose meaning is "one or the other or both." The **exclusive or,** meaning "one or the other but not both" and denoted $\circledvee$, has its uses in English, as in "She will marry Heckle or she will marry Jeckle." The "inclusive or" is much more useful in mathematics and is the accepted meaning unless there is a statement to the contrary.
★ (i) Make a truth table for the "exclusive or" connective $\circledvee$.
(ii) Show that $A \circledvee B$ is equivalent to $(A \vee B) \wedge \sim(A \wedge B)$.
(b) "NAND" and "NOR" circuits are commonly used as a basis for flash memory chips. A NAND B is defined to be the negation of "A and B." A NOR B is defined to be the negation of "A or B."
(i) Write truth tables for NAND and NOR connectives.
(ii) Show that (A NAND B) $\vee$ (A NOR B) is equivalent to (A NAND B).
(iii) Show that (A NAND B) $\wedge$ (A NOR B) is equivalent to (A NOR B).

1.2 Conditionals and Biconditionals

Sentences of the form "If P, then Q" are the most important kind of propositions in mathematics. You have seen many examples of such statements in mathematics courses: from precalculus, "If two lines in a plane have the same slope, then the lines are parallel"; from trigonometry, "If $\sec\theta = \frac{5}{3}$, then $\sin\theta = \frac{4}{5}$."; from calculus, "If f is differentiable at x_0 and $f(x_0)$ is a relative minimum for f, then $f'(x_0) = 0$."

> **DEFINITIONS** For propositions P and Q, the **conditional sentence** $P \Rightarrow Q$ is the proposition "If P, then Q." Proposition P is called the **antecedent** and Q is the **consequent.** The conditional sentence $P \Rightarrow Q$ is true if and only if P is false or Q is true.