roses are red, nor that violets are blue" in at least two ways: $\sim(P \vee Q)$ or $(\sim P) \wedge (\sim Q)$. Fortunately, these are equivalent by Theorem 1.1.1(h). Note that the proposition "Violets are purple" requires a new symbol, say R, since it expresses a new idea that cannot be formed from the components P and Q.

The sentence "17 and 35 have no common divisors" shows that the meaning, and not just the form of the sentence, must be considered in translating; it cannot be broken up into the two propositions: "17 has no common divisors" and "35 has no common divisors." Compare this with the proposition "17 and 35 have digits totaling 8," which can be written as a conjunction.

Example. Suppose b is a fixed real number. The form of the sentence "If b is an integer, then b is either even or odd" is $P \Rightarrow (Q \vee R)$, where P is "b is an integer," Q is "b is even," and R is "b is odd."

Example. Suppose a, b, and p are fixed integers. "If p is a prime number that divides ab, then p divides a or b" has the form $(P \wedge Q) \Rightarrow (R \vee S)$, where P is "p is a prime," Q is "p divides ab," R is "p divides a," and S is "p divides b."

The hierarchy of connectives in Section 1.1 that governs the use of parentheses for propositional forms can be extended to the connectives $\Rightarrow$ and $\Leftrightarrow$:

The connectives $\sim$, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are always applied in the order listed.

Thus, $\sim$ applies to the smallest possible proposition, then $\wedge$ is applied with the next smallest scope, and so forth. For example,

$$P \Rightarrow \sim Q \vee R \Leftrightarrow S \text{ is an abbreviation for } (P \Rightarrow [(\sim Q) \vee R]) \Leftrightarrow S,$$
$$P \vee \sim Q \Leftrightarrow R \Rightarrow S \text{ is an abbreviation for } [P \vee (\sim Q)] \Leftrightarrow (R \Rightarrow S),$$

and

$$P \Rightarrow Q \Rightarrow R \text{ is an abbreviation for } (P \Rightarrow Q) \Rightarrow R.$$

Exercises 1.2

1. Identify the antecedent and the consequent for each of the following conditional sentences. Assume that a, b, and f represent some fixed sequence, integer, or function, respectively.

★ **(a)** If squares have three sides, then triangles have four sides.
(b) If the moon is made of cheese, then 8 is an irrational number.
(c) b divides 3 only if b divides 9.
★ **(d)** The differentiability of f is sufficient for f to be continuous.
(e) A sequence a is bounded whenever a is convergent.
★ **(f)** A function f is bounded if f is integrable.
(g) $1 + 2 = 3$ is necessary for $1 + 1 = 2$.

(h) The fish bite only when the moon is full.

★ **(i)** A time of 3 minutes, 48 seconds or less is necessary to qualify for the Olympic team.

☆ **2.** Write the converse and contrapositive of each conditional sentence in Exercise 1.

3. What can be said about the truth value of Q when

(a) P is false and $P \Rightarrow Q$ is true? **(b)** P is true and $P \Rightarrow Q$ is true?
(c) P is true and $P \Rightarrow Q$ is false? **(d)** P is false and $P \Leftrightarrow Q$ is true?
(e) P is true and $P \Leftrightarrow Q$ is false?

4. Identify the antecedent and consequent for each conditional sentence in the following statements from this book.

(a) Theorem 1.3.1(a) **(b)** Exercise 3 of Section 1.6
(c) Theorem 2.1.4 **(d)** The PMI, Section 2.4
(e) Theorem 2.6.4 **(f)** Theorem 3.4.2
(g) Theorem 4.2.2 **(h)** Theorem 5.1.7(a)

5. Which of the following conditional sentences are true?

★ **(a)** If triangles have three sides, then squares have four sides.
(b) If a hexagon has six sides, then the moon is made of cheese.
★ **(c)** If $7 + 6 = 14$, then $5 + 5 = 10$.
(d) If $5 < 2$, then $10 < 7$.
★ **(e)** If one interior angle of a right triangle is $92°$, then the other interior angle is $88°$.
(f) If Euclid's birthday was April 2, then rectangles have four sides.
(g) 5 is prime if $\sqrt{2}$ is not irrational.
(h) $1 + 1 = 2$ is sufficient for $3 > 6$.

6. Which of the following are true?

★ **(a)** Triangles have three sides iff squares have four sides.
(b) $7 + 5 = 12$ iff $1 + 1 = 2$.
★ **(c)** b is even iff $b + 1$ is odd. (Assume that b is some fixed integer.)
(d) m is odd iff m^2 is odd. (Assume that m is some fixed integer.)
(e) $5 + 6 = 6 + 5$ iff $7 + 1 = 10$.
(f) A parallelogram has three sides iff 27 is prime.
(g) The Eiffel Tower is in Paris if and only if the chemical symbol for helium is H.
(h) $\sqrt{10} + \sqrt{13} < \sqrt{11} + \sqrt{12}$ iff $\sqrt{13} - \sqrt{12} < \sqrt{11} - \sqrt{10}$.
(i) $x^2 \geq 0$ iff $x \geq 0$. (Assume that x is a fixed real number.)
(j) $x^2 - y^2 = 0$ iff $(x - y)(x + y) = 0$. (Assume that x and y are fixed real numbers.)
(k) $x^2 + y^2 = 50$ iff $(x + y)^2 = 50$. (Assume that x and y are fixed real numbers.)

7. Make truth tables for these propositional forms.

(a) $P \Rightarrow (Q \wedge P)$. ★ **(b)** $(\sim P \Rightarrow Q) \vee (Q \Leftrightarrow P)$.
★ **(c)** $\sim Q \Rightarrow (Q \Leftrightarrow P)$. **(d)** $(P \vee Q) \Rightarrow (P \wedge Q)$.
(e) $(P \wedge Q) \vee (Q \wedge R) \Rightarrow P \vee R$.
(f) $[(Q \Rightarrow S) \wedge (Q \Rightarrow R)] \Rightarrow [(P \vee Q) \Rightarrow (S \vee R)]$.

8. Prove Theorem 1.2.2 by constructing truth tables for each equivalence.

9. Determine whether each statement qualifies as a definition.

(a) $y = f(x)$ is a linear function when its graph is a straight line.

(b) $y = f(x)$ is a quadratic function when it contains an x^2 term.

(c) m is a perfect square when $m = n^2$ for some integer n.

(d) A triangle is a right triangle when the sum of two of its interior angles is 90°.

(e) Two lines are parallel when their slopes are the same number.

(f) A sundial is an instrument for measuring time.

10. Rewrite each of the following sentences using logical connectives. Assume that each symbol f, x_0, n, x, S, $\mathbf{B}$ represents some fixed object.

★ **(a)** If f has a relative minimum at x_0 and if f is differentiable at x_0, then $f'(x_0) = 0$.

(b) If n is prime, then $n = 2$ or n is odd.

(c) A number x is real and not rational whenever x is irrational.

★ **(d)** If $x = 1$ or $x = -1$, then $|x| = 1$.

★ **(e)** f has a critical point at x_0 iff $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

(f) S is compact iff S is closed and bounded.

(g) $\mathbf{B}$ is invertible is a necessary and sufficient condition for det $\mathbf{B} \neq 0$.

(h) $6 \geq n - 3$ only if $n > 4$ or $n > 10$.

(i) x is Cauchy implies x is convergent.

(j) f is continuous at x_0 whenever $\lim_{x \to x_0} f(x) = f(x_0)$.

(k) If f is differentiable at x_0 and f is strictly increasing at x_0, then $f'(x_0) > 0$.

11. Dictionaries indicate that the conditional meaning of *unless* is preferred, but there are other interpretations as a converse or a biconditional. Discuss the translation of each sentence.

(a) I will go to the store unless it is raining.

★ **(b)** The Dolphins will not make the playoffs unless the Bears win all the rest of their games.

(c) You cannot go to the game unless you do your homework first.

(d) You won't win the lottery unless you buy a ticket.

12. Show that the following pairs of statements are equivalent.

(a) $(P \vee Q) \Rightarrow R$ and $\sim R \Rightarrow (\sim P \wedge \sim Q)$.

★ **(b)** $(P \wedge Q) \Rightarrow R$ and $(P \wedge \sim R) \Rightarrow \sim Q$.

(c) $P \Rightarrow (Q \wedge R)$ and $(\sim Q \vee \sim R) \Rightarrow \sim P$.

(d) $P \Rightarrow (Q \vee R)$ and $(P \wedge \sim R) \Rightarrow Q$.

(e) $(P \Rightarrow Q) \Rightarrow R$ and $(P \wedge \sim Q) \vee R$.

(f) $P \Leftrightarrow Q$ and $(\sim P \vee Q) \wedge (\sim Q \vee P)$.

13. Give, if possible, an example of a true conditional sentence for which

★ **(a)** the converse is true.
(b) the converse is false.

★ **(c)** the contrapositive is false.
(d) the contrapositive is true.

14. Give, if possible, an example of a false conditional sentence for which

(a) the converse is true.
(b) the converse is false.

(c) the contrapositive is false.
(d) the contrapositive is true.

15. Give the converse and contrapositive of each sentence of Exercises 10(a), (b), (c), and (d). Tell whether each converse and contrapositive is true or false.

16. Determine whether each of the following is a tautology, a contradiction, or neither.

★ **(a)** $[(P \Rightarrow Q) \Rightarrow P] \Rightarrow P$.
(b) $P \Leftrightarrow P \wedge (P \vee Q)$.
(c) $P \Rightarrow Q \Leftrightarrow P \wedge {\sim}Q$.
★ **(d)** $P \Rightarrow [P \Rightarrow (P \Rightarrow Q)]$.
(e) $P \wedge (Q \vee {\sim}Q) \Leftrightarrow P$.
(f) $[Q \wedge (P \Rightarrow Q)] \Rightarrow P$.
(g) $(P \Leftrightarrow Q) \Leftrightarrow {\sim}({\sim}P \vee Q) \vee ({\sim}P \wedge Q)$.
(h) $[P \Rightarrow (Q \vee R)] \Rightarrow [(Q \Rightarrow R) \vee (R \Rightarrow P)]$.
(i) $P \wedge (P \Leftrightarrow Q) \wedge {\sim}Q$.
(j) $(P \vee Q) \Rightarrow Q \Rightarrow P$.
(k) $[P \Rightarrow (Q \wedge R)] \Rightarrow [R \Rightarrow (P \Rightarrow Q)]$.
(l) $[P \Rightarrow (Q \wedge R)] \Rightarrow R \Rightarrow (P \Rightarrow Q)$.

17. The **inverse,** or **opposite,** of the conditional sentence $P \Rightarrow Q$ is ${\sim}P \Rightarrow {\sim}Q$.

(a) Show that $P \Rightarrow Q$ and its inverse are not equivalent forms.
(b) For what values of the propositions P and Q are $P \Rightarrow Q$ and its inverse both true?
(c) Which is equivalent to the converse of a conditional sentence, the contrapositive of its inverse, or the inverse of its contrapositive?

1.3 Quantifiers

Unless there has been a prior agreement about the value of x, the statement "$x \geq 3$" is neither true nor false. A sentence that contains variables is called an **open sentence** or **predicate**, and becomes a proposition only when its variables are assigned specific values. For example, "$x \geq 3$" is true when x is given the value 7 and false when $x = 2$.

When P is an open sentence with a variable x, the sentence is symbolized by $P(x)$. Likewise, if P has variables $x_1, x_2, x_3, \ldots, x_n$, the sentence may be denoted by $P(x_1, x_2, x_3, \ldots, x_n)$. For example, for the sentence "$x + y = 3z$" we write $P(x, y, z)$, and we see that $P(4, 5, 3)$ is true because $4 + 5 = 3(3)$, while $P(1, 2, 4)$ is false.

The collection of objects that may be substituted to make an open sentence a true proposition is called the **truth set** of the sentence. Before a truth set can be determined, we must be given or must decide what objects are available for consideration; that is, we must have specified a **universe of discourse.** In many cases the universe will be understood from the context. For a sentence such as "x likes chocolate," the universe is presumably the set of all people. We will often use the number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ as our universes. (See the *Preface to the Student*.)

Example. The truth set of the open sentence "$x^2 < 5$" depends upon the collection of objects we choose for the universe of discourse. With the universe specified as the set $\mathbb{N}$, the truth set is $\{1, 2\}$. For the universe $\mathbb{Z}$, the truth set is $\{-2, -1, 0, 1, 2\}$. When the universe is $\mathbb{R}$, the truth set is the open interval $(-\sqrt{5}, \sqrt{5})$.