Recall that for $(\exists x)P(x)$ to be true it is unimportant how many elements are in the truth set of $P(x)$, as long as there is at least one. For $(\exists!x)P(x)$ to be true, the number of elements in the truth set of $P(x)$ is crucial—there must be exactly one.

In the universe of natural numbers, $(\exists!x)$ (x is even and x is prime) is true because the truth set of "x is even and x is prime" contains only the number 2. The sentence $(\exists!x)(x^2 = 4)$ is true in the universe of natural numbers, but false in the universe of all integers.

Theorem 1.3.2 If $A(x)$ is an open sentence with variable x, then

(a) $(\exists!x)A(x) \Rightarrow (\exists x)A(x)$.

(b) $(\exists!x)A(x)$ is equivalent to $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \Rightarrow y = z)$.

Part (a) of Theorem 1.3.2 says that $\exists!$ is indeed a special case of the quantifier $\exists$. Part (b) says that "There exists a unique x such that $A(x)$" is equivalent to "There is an x such that $A(x)$ and if both $A(y)$ and $A(z)$, then $y = z$." The proofs are left to Exercise 11.

Exercises 1.3

1. Translate the following English sentences into symbolic sentences with quantifiers. The universe for each is given in parentheses.

★ **(a)** Not all precious stones are beautiful. (All stones)

☆ **(b)** All precious stones are not beautiful. (All stones)

(c) Some isosceles triangle is a right triangle. (All triangles)

(d) No right triangle is isosceles. (All triangles)

(e) All people are honest or no one is honest. (All people)

(f) Some people are honest and some people are not honest. (All people)

(g) Every nonzero real number is positive or negative. (Real numbers)

★ **(h)** Every integer is greater than -4 or less than 6. (Real numbers)

(i) Every integer is greater than some integer. (Integers)

★ **(j)** No integer is greater than every other integer. (Integers)

(k) Between any integer and any larger integer, there is a real number. (Real numbers)

★ **(l)** There is a smallest positive integer. (Real numbers)

★ **(m)** No one loves everybody. (All people)

(n) Everybody loves someone. (All people)

(o) For every positive real number x, there is a unique real number y such that $2^y = x$. (Real numbers)

☆ **2.** For each of the propositions in Exercise 1, write a useful denial, and give a translation into ordinary English.

3. Translate these definitions from the *Preface to the Student* into quantified sentences.

(a) The integer x is **even**.

(b) The integer x is **odd**.

(c) The integer a **divides** the integer b.
(d) The natural number n is **prime**.
(e) The natural number n is **composite**.

4. Translate these definitions in this text into quantified sentences. You need not know the specifics of the terms and symbols to complete this exercise.
   (a) The relation R is **symmetric**. (See page 147.)
   (b) The relation R is **transitive**. (See page 147.)
   (c) The function f is **one-to-one**. (See page 208.)
   (d) The operation $*$ is **commutative**. (See page 277.)

☆ 5. The sentence "People dislike taxes" might be interpreted to mean "All people dislike all taxes," "All people dislike some taxes," "Some people dislike all taxes," or "Some people dislike some taxes." Give a symbolic translation for each of these interpretations.

6. Let $T = \{17\}$, $U = \{6\}$, $V = \{24\}$, and $W = \{2, 3, 7, 26\}$. In which of these four different universes is the statement true?
   ★ a) $(\exists x)(x \text{ is odd} \Rightarrow x > 8)$.
   b) $(\exists x)(x \text{ is odd} \wedge x > 8)$.
   c) $(\forall x)(x \text{ is odd} \Rightarrow x > 8)$.
   d) $(\forall x)(x \text{ is odd} \wedge x > 8)$.

7. (a) Complete this proof of Theorem 1.3.1(b):
   Proof: Let U be any universe.
   The sentence $\sim(\exists x)A(x)$ is true in U
   iff . . .
   iff $(\forall x)\sim A(x)$ is true in U.
   ☆ (b) Give a proof of part (b) of Theorem 1.3.1 that uses part (a).

8. Which of the following are true? The universe for each statement is given in parentheses.
   (a) $(\forall x)(x + x \geq x)$. $(\mathbb{R})$
   ★ (b) $(\forall x)(x + x \geq x)$. $(\mathbb{N})$
   (c) $(\exists x)(2x + 3 = 6x + 7)$. $(\mathbb{N})$
   (d) $(\exists x)(3^x = x^2)$. $(\mathbb{R})$
   ★ (e) $(\exists x)(3^x = x)$. $(\mathbb{R})$
   (f) $(\exists x)(3(2 - x) = 5 + 8(1 - x))$. $(\mathbb{R})$
   (g) $(\forall x)(x^2 + 6x + 5 \geq 0)$. $(\mathbb{R})$
   ★ (h) $(\forall x)(x^2 + 4x + 5 \geq 0)$. $(\mathbb{R})$
   (i) $(\exists x)(x^2 - x + 41 \text{ is prime})$. $(\mathbb{N})$
   (j) $(\forall x)(x^2 - x + 41 \text{ is prime})$. $(\mathbb{N})$
   (k) $(\forall x)(x^3 + 17x^2 + 6x + 100 \geq 0)$. $(\mathbb{R})$
   (l) $(\forall x)(\forall y)[x < y \Rightarrow (\exists w)(x < w < y)]$. $(\mathbb{Q})$

9. Give an English translation for each. The universe is given in parentheses.
   (a) $(\forall x)(x \geq 1)$. $(\mathbb{N})$
   ★ (b) $(\exists! x)(x \geq 0 \wedge x \leq 0)$. $(\mathbb{R})$
   (c) $(\forall x)(x \text{ is prime} \wedge x \neq 2 \Rightarrow x \text{ is odd})$. $(\mathbb{N})$
   ★ (d) $(\exists! x)(\log_e x = 1)$. $(\mathbb{R})$

(e) $\sim(\exists x)(x^2 < 0)$. ($\mathbb{R}$)
(f) $(\exists! x)(x^2 = 0)$. ($\mathbb{R}$)
(g) $(\forall x)(x \text{ is odd} \Rightarrow x^2 \text{ is odd})$. ($\mathbb{N}$)

10. Which of the following are true in the universe of all real numbers?
★ **(a)** $(\forall x)(\exists y)(x + y = 0)$.
(b) $(\exists x)(\forall y)(x + y = 0)$.
(c) $(\exists x)(\exists y)(x^2 + y^2 = -1)$.
★ **(d)** $(\forall x)[x > 0 \Rightarrow (\exists y)(y < 0 \wedge xy > 0)]$.
(e) $(\forall y)(\exists x)(\forall z)(xy = xz)$.
★ **(f)** $(\exists x)(\forall y)(x \leq y)$.
(g) $(\forall y)(\exists x)(x \leq y)$.
(h) $(\exists! y)(y < 0 \wedge y + 3 > 0)$.
★ **(i)** $(\exists! x)(\forall y)(x = y^2)$.
(j) $(\forall y)(\exists! x)(x = y^2)$.
(k) $(\exists! x)(\exists! y)(\forall w)(w^2 > x - y)$.

11. Let $A(x)$ be an open sentence with variable x.
☆ **(a)** Prove Theorem 1.3.2 (a).
☆ **(b)** Show that the converse of Theorem 1.3.2 (a) is false.
(c) Prove Theorem 1.3.2 (b).
(d) Prove that $(\exists! x)A(x)$ is equivalent to $(\exists x)[A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)]$.
★ **(e)** Find a useful denial for $(\exists! x)A(x)$.

12. **(a)** Write the symbolic form for the definition of "f is continuous at a."
(b) Write the symbolic form of the statement of the Mean Value Theorem.
(c) Write the symbolic form for the definition of "$\lim_{x \to a} f(x) = L$."
(d) Write a useful denial of each sentence in parts (a), (b), and (c).

13. Which of the following are denials of $(\exists! x)P(x)$?
(a) $(\forall x)P(x) \vee (\forall x)\sim P(x)$.
(b) $(\forall x)\sim P(x) \vee (\exists y)(\exists z)(y \neq z \wedge P(y) \wedge P(z))$.
(c) $(\forall x)[P(x) \Rightarrow (\exists y)(P(y) \wedge x \neq y)]$.
★ **(d)** $\sim(\forall x)(\forall y)[(P(x) \wedge P(y)) \Rightarrow x = y]$.

★ **14.** *Riddle:* What is the English translation of the symbolic statement $\forall\exists\exists\forall$?

1.4 Basic Proof Methods I

In mathematics, a **theorem** is a statement that describes a pattern or relationship among quantities or structures and a **proof** is a justification of the truth of a theorem. Before beginning to examine valid proof techniques it is recommended that you review the comments about proofs and the definitions in the *Preface to the Student*.

We cannot define all terms nor prove all statements from previous ones. We begin with an initial set of statements, called **axioms** (or **postulates**), that are *assumed to be true*. We then derive theorems that are true in any situation where the