Composites and inverses of permutations can be easily computed using the notation described above. Remember to evaluate a composite from right to left, because the function on the right is applied first.

Examples. For $A = \{1, 2, 3, 4\}$ and the functions $t = [2\ 4\ 3\ 1]$ and $s = [2\ 3\ 1\ 4]$, $t \circ s$ is $[2\ 4\ 3\ 1] \circ [2\ 3\ 1\ 4] = [4\ 3\ 2\ 1]$. The thought process begins with $[2\ 3\ 1\ 4]$, where "1 goes to 2." Then we see that in $[2\ 4\ 1\ 3]$, "2 goes to 4." Thus in the composite "1 goes to 4." We find the other images similarly. For example, s sends 2 to 3 and t sends 3 to 3, so the composite sends 2 to 3.

To find the inverse of the permutation $t = [2\ 4\ 3\ 1]$, we think of reversing its action. The permutation t sends 1 to 2 so the inverse of t sends 2 to 1 Since t sends 2 to 4, t^{-1} sends 4 to 2, etc. In this way we find that $t^{-1} = [4\ 1\ 3\ 2]$. We can verify that this is correct by computing $[2\ 4\ 3\ 1] \circ [4\ 1\ 3\ 2] = [1\ 2\ 3\ 4]$, which is the identity permutation.

Exercises 4.4

1. Show that each of these functions is a one-to-one correspondence.

(a) $f\colon (2, \infty) \to (-\infty, -1)$ given by $f(x) = \dfrac{-x}{x-2}$.

(b) $g\colon (-\infty, -4) \to (-\infty, 0)$ given by $g(x) = -|x + 4|$.

(c) $h\colon 8\mathbb{N} \to 10\mathbb{N}$, where $8\mathbb{N} = \{8k\colon k \in \mathbb{N}\}$, $10\mathbb{N} = \{10k\colon k \in \mathbb{N}\}$, and $h(x) = 1.25x$.

(d) $G\colon \mathbb{N} \times \mathbb{N} \to 8\mathbb{N}$ given by $G(m, n) = 2^{m+2}(2n - 1)$.

(e) $k\colon \mathbb{R} \to (3, \infty)$ given by $k(x) = 2e^x + 3$. You may use the fact that an exponential function maps $\mathbb{R}$ one-to-one and onto $(0, \infty)$.

2. Find a one-to-one correspondence between each of these pairs of sets. Prove that your function is one-to-one and onto the given codomain.

(a) $\{a, b, c, d, e, f\}$ and $\{2, 4, 8, 16, 32, 64\}$

(b) $\mathbb{N}$ and $\mathbb{N} - \{1\}$

(c) $(3, \infty)$ and $(5, \infty)$

(d) $(-\infty, 1)$ and $(-1, \infty)$

(e) $12\mathbb{N}$ and $20\mathbb{N}$, where $12\mathbb{N} = \{12k\colon k \in \mathbb{N}\}$ and $20\mathbb{N} = \{20k\colon k \in \mathbb{N}\}$

3. For each one-to-one-correspondence, find the inverse function. Verify your answer by computing the composite of the function and its inverse.

(a) $f\colon (0, \infty) \to (0, \infty)$ given by $f(x) = \dfrac{1}{x}$.

★ **(b)** $g\colon (-2, \infty) \to (-\infty, 4)$ given by $g(x) = \dfrac{4x}{x+2}$.

(c) $h\colon \mathbb{R} \to (0, \infty)$ given by $h(x) = e^{x+3}$.

(d) $G\colon (3, \infty) \to (5, \infty)$ given by $G(x) = \dfrac{5(x-1)}{x-3}$.

4. Prove part (b) of Theorem 4.4.2: If $F\colon A \to B$ and F^{-1} is a function, then F is one-to-one.

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5. **(a)** Assume that $F: A \xrightarrow[\text{onto}]{1-1} B$. Prove that $G = F^{-1}$ iff $G \circ F = I_A$ or $F \circ G = I_B$ (Theorem 4.4.4, part (b)).

(b) Give an example of sets A and B and functions F and G such that $F: A \to B$, $G: B \to A$, $G \circ F = I_A$ and $G \neq F^{-1}$.

6. Let $F: A \to B$ and $G: B \to A$. Use the results of this section to prove that if $G \circ F = I_A$ and $F \circ G = I_B$, then $F: A \xrightarrow[\text{onto}]{1-1} B$ and $G: B \xrightarrow[\text{onto}]{1-1} A$.

7. Use the one-to-one correspondences $\ln: (0, \infty) \to \mathbb{R}$ and $f: (2, \infty) \to (0, \infty)$, where $f(x) = x - 2$, to describe a one-to-one correspondence

(a) from $(0, \infty)$ onto $(2, \infty)$.

(b) from $(2, \infty)$ onto $\mathbb{R}$.

(c) from $\mathbb{R}$ onto $(2, \infty)$.

8. Prove that if $f: A \xrightarrow[\text{onto}]{1-1} B$, $g: B \xrightarrow[\text{onto}]{1-1} C$, and $h: C \xrightarrow[\text{onto}]{1-1} D$, then $f^{-1} \circ g^{-1} \circ h^{-1}$ is a one-to-one correspondence from D to A.

9. Use the notation of this section to write these permutations of the set $C = \{1, 2, 3, 4, 5, 6, 7\}$.

(a) I_C **(b)** $u = \{(1, 5), (6, 7), (4, 4), (5, 1), (3, 2), (2, 6), (7, 3)\}$

★ **(c)** $v = \{(1, 2), (2, 5), (3, 4), (5, 1), (4, 6), (6, 7), (7, 3)\}$

(d) $w = \{(1, 2), (2, 1), (3, 4), (4, 3), (5, 5), (6, 7), (7, 6)\}$

★ **(e)** $u \circ v$ **(f)** $v \circ u$ **(g)** $w \circ w$ **(h)** $u \circ v \circ w$ ★ **(i)** u^{-1}

(j) v^{-1} **(k)** $(u \circ v)^{-1}$ **(l)** $v^{-1} \circ u^{-1}$ **(m)** $u^{-1} \circ v^{-1}$

Proofs to Grade

10. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

(a) **Claim.** If f is a one-to-one correspondence from A to B and g is a one-to-one correspondence from B to A, then $g = f^{-1}$.

"Proof." Suppose f is a one-to-one correspondence from A to B and g is a one-to-one correspondence from B to A. Then by Theorem 4.4.1, $g \circ f$ is a one-to-one correspondence from A to A. Likewise, $f \circ g$ is a one-to-one correspondence from A to A. Then $f^{-1} = f^{-1} \circ (g \circ f) = f^{-1} \circ (f \circ g) = (f^{-1} \circ f) \circ g = I_A \circ g = g$. ■

(b) **Claim.** If f is a permutation of A, then $I_A \circ f = f \circ I_A$.

"Proof." Since I_A is the identity, $I_A \circ f = f$. Also I_A is the identity, so $f \circ I_A = f$. Therefore $I_A \circ f = f \circ I_A$. ■

(c) **Claim.** Let $r = [1\ 2\ 4\ 3\ 5]$ and $s = [4\ 2\ 3\ 1\ 5]$ be permutations of $\{1, 2, 3, 4, 5\}$. Then the inverse of $r \circ s$ is $[3\ 2\ 4\ 1\ 5]$.

"Proof." Let $r = [1\ 2\ 4\ 3\ 5]$ and $s = [4\ 2\ 3\ 1\ 5]$. Then $r^{-1} = r$ and $s^{-1} = s$. Therefore $(r \circ s)^{-1} = r^{-1} \circ s^{-1} = r \circ s = [3\ 2\ 4\ 1\ 5]$. ■

(d) **Claim.** If f and g are permutations of A, then $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

"Proof." We know by Theorem 4.4.5 that $f \circ g$ is a permutation of A, so $(f \circ g)^{-1}$ is a permutation of A. Also by Theorem 4.4.5, g^{-1} and f^{-1} are permutations of A, so $g^{-1} \circ f^{-1}$ is a permutation of A. By Theorem 4.4.4, we can check whether $g^{-1} \circ f^{-1}$ is the inverse of $f \circ g$ by computing their composite. We find that $(f \circ g) \circ (g^{-1} \circ f^{-1}) = f \circ (g \circ (g^{-1} \circ f^{-1})) = f \circ (g \circ g^{-1}) \circ f^{-1} = f \circ I_A \circ f^{-1} = (f \circ I_A) \circ f^{-1} = f \circ f^{-1} = I_A$. Therefore $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. ■

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