Proof.

(a) Suppose $b \in f(C \cap D)$. Then $b = f(a)$ for some $a \in C \cap D$. Since $a \in C$ and $b = f(a)$, $b \in f(C)$. Also, $a \in D$ and $b = f(a)$, so $b \in f(D)$. Thus, $b \in f(C) \cap f(D)$.

(c) $a \in f^{-1}(E \cap F)$
iff $f(a) \in E \cap F$
iff $f(a) \in E$ and $f(a) \in F$
iff $a \in f^{-1}(E)$ and $a \in f^{-1}(F)$
iff $a \in f^{-1}(E) \cap f^{-1}(F)$.

The proofs of parts (b) and (d) are left as exercises. ■

Finally, we note that if $f: A \to B$, then every subset X of A has a corresponding image set $f(X)$ that is a subset of B. This correspondence is a function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ called the **induced function**, $f: \mathcal{P}(A) \to \mathcal{P}(B)$. Note that the same letter, f, is used for the function and the induced function. The appropriate interpretation is usually clear from the context.

Likewise, every subset $Y \subseteq B$ has a unique inverse image $f^{-1}(Y)$ that is a subset of A. This correspondence is another induced function, $f^{-1}: \mathcal{P}(B) \to \mathcal{P}(A)$.

Exercises 4.5

1. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $h = \{(1, 4), (2, 4), (3, 5)\}$.

★ **(a)** Find the image of each of the 8 subsets of A.

(b) Find the inverse image of each of the 8 subsets of B.

2. Let $f(x) = x^2 + 1$. Find

★ **(a)** $f([1, 3])$.

(b) $f([-1, 0] \cup [2, 4])$.

★ **(c)** $f^{-1}([-1, 1])$.

(d) $f^{-1}([-2, 3])$.

(e) $f^{-1}([5, 10])$.

(f) $f^{-1}([-1, 5] \cup [17, 26])$.

3. Let $f(x) = 1 - 2x$. Find

(a) $f(A)$ where $A = \{-1, 0, 1, 2, 3\}$.

(b) $f(\mathbb{N})$.

(c) $f^{-1}(\mathbb{R})$.

(d) $f^{-1}([2, 5])$.

(e) $f((1, 4])$.

(f) $f(f^{-1}(f([3, 4]))$.

4. Let $f: \mathbb{R} - \{0\} \to \mathbb{R}$ be given by $f(x) = x + \dfrac{1}{x}$. Find

(a) $f((0, 2))$.

★ **(b)** $f([1, 5])$.

★ **(c)** $f^{-1}((3, 4])$.

(d) $f^{-1}([0, 1))$.

(e) $f(f^{-1}(\mathbb{R}))$.

(f) $f\left(f^{-1}\left(\left[\frac{-10}{3}, 10.1\right]\right)\right)$

5. Let $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $f(m, n) = 2^m 3^n$. Find

(a) $f(A \times B)$ where $A = \{1, 2, 3\}$, $B = \{3, 4\}$.

(b) $f^{-1}(\{5, 6, 7, 8, 9, 10\})$.

6. Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ 2 - 2x & \text{if } x < 1 \end{cases}$. Give an example of

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(a) $a \in \mathbb{R}$ and $D \subseteq \mathbb{R}$ such that $f(a) \in f(D)$ and $a \notin D$.
(b) subsets A and C of $\mathbb{R}$ such that $f(A \cap C) \neq f(A) \cap f(C)$.
(c) a subset D of $\mathbb{R}$ such that $D \neq f^{-1}(f(D))$.
(d) a subset E of $\mathbb{R}$ such that $E \neq f(f^{-1}(E))$.

7. Prove parts (b) and (d) of Theorem 4.5.1.

8. Let $f: A \to B$, and let $\{D_\alpha: \alpha \in \Delta\}$ and $\{E_\beta: \beta \in \Gamma\}$ be families of subsets of A and B, respectively. Prove that

★ **(a)** $f\left(\bigcap_{\alpha\in\Delta} D_\alpha\right) \subseteq \bigcap_{\alpha\in\Delta} f(D_\alpha)$.

(b) $f\left(\bigcup_{\alpha\in\Delta} D_\alpha\right) = \bigcup_{\alpha\in\Delta} f(D_\alpha)$.

(c) $f^{-1}\left(\bigcap_{\beta\in\Gamma} E_\beta\right) = \bigcap_{\beta\in\Gamma} f^{-1}(E_\beta)$.

(d) $f^{-1}\left(\bigcup_{\beta\in\Gamma} E_\beta\right) = \bigcup_{\beta\in\Gamma} f^{-1}(E_\beta)$.

☆ **9.** Give an example of a function $f: \mathbb{R} \to \mathbb{R}$, and a family $\{D_\alpha: \alpha \in \Delta\}$ of subsets of $\mathbb{R}$ such that $f\left(\bigcap_{\alpha\in\Delta} D_\alpha\right) \neq \bigcap_{\alpha\in\Delta} f(D_\alpha)$.

10. Let $f: A \to B$, $D \subseteq A$, and $E \subseteq B$. Prove that

★ **(a)** $f(f^{-1}(E)) \subseteq E$.
(b) $A - f^{-1}(E) \subseteq f^{-1}(B - E)$.
(c) $f^{-1}(B - E) \subseteq A - f^{-1}(E)$.
★ **(d)** $E = f(f^{-1}(E))$ iff $E \subseteq \text{Rng}(f)$.
(e) $D \subseteq f^{-1}(f(D))$.
(f) $D = f^{-1}(f(D))$ iff $f(A - D) \subseteq B - f(D)$.

11. Let $f: A \to B$ and let $X, Y \subseteq A$ and $U, V \subseteq B$. Prove that

(a) $f(X) \subseteq U$ iff $X \subseteq f^{-1}(U)$.
★ **(b)** $f(X) - f(Y) \subseteq f(X - Y)$.
(c) $f^{-1}(U) - f^{-1}(V) = f^{-1}(U - V)$.

☆ **12.** Let $f: A \to B$. Prove that if f is one-to-one, then $f(X) \cap f(Y) = f(X \cap Y)$ for all $X, Y \subseteq A$. Is the converse true? Explain.

13. Let $f: A \to B$. Prove that if $X \subseteq A$ and f is one-to-one, then $f(A - X) = f(A) - f(X)$.

14. Let $f: A \to B$. Prove that if $X \subseteq A$, $Y \subseteq B$, and f is a bijection, then $f(X) = Y$ iff $f^{-1}(Y) = X$.

15. Let $f: A \to B$.

★ **(a)** What condition on f will ensure that the induced function $f: \mathcal{P}(A) \to \mathcal{P}(B)$ is one-to-one?

(b) What condition on f will ensure that the induced function $f: \mathcal{P}(A) \to \mathcal{P}(B)$ is onto $\mathcal{P}(B)$?

16. Let $f: A \to B$ and $K \subseteq B$. Prove that $f(f^{-1}(K)) = K \cap \text{Rng}(f)$.

17. Let $f: A \to B$. Let T be the relation on A defined by $x\,T\,y$ iff $f(x) = f(y)$. By Exercise 18 (a) of Section 4.1, T is an equivalence relation on A. Describe the partition of A associated with T.

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Proofs to Grade

18. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

★ **(a)** **Claim.** If $f: A \to B$ and $X \subseteq A$, then $f^{-1}(f(X)) \subseteq X$.
"Proof." If $x \in f^{-1}(f(X))$, then by definition of f^{-1}, $f(x) \in f(X)$. Therefore $x \in X$. Thus $f^{-1}(f(X)) \subseteq X$. ■

(b) **Claim.** If $f: A \to B$ and $X \subseteq A$, then $X \subseteq f^{-1}(f(X))$.
"Proof." Suppose $z \in X$. Then $f(z) \in f(X)$. Therefore we conclude that $z \in f^{-1}(f(X))$, which proves the set inclusion. ■

(c) **Claim.** If $f: A \to B$ and $\{D_\alpha: \alpha \in \Delta\}$ is a family of subsets of A, then $\bigcap_{\alpha\in\Delta} f(D_\alpha) \subseteq f\left(\bigcap_{\alpha\in\Delta} D_\alpha\right)$.

"Proof." Suppose $y \in \bigcap_{\alpha\in\Delta} f(D_\alpha)$. Then $y \in f(D_\alpha)$ for all α. Thus there exists $x \in D_\alpha$ such that $f(x) = y$, for all α. Then $x \in \bigcap_{\alpha\in\Delta} D_\alpha$ and $f(x) = y$, so $y \in f\left(\bigcap_{\alpha\in\Delta} D_\alpha\right)$. Therefore, $\bigcap_{\alpha\in\Delta} f(D_\alpha) \subseteq f\left(\bigcap_{\alpha\in\Delta} D_\alpha\right)$. ■

4.6 Sequences

In calculus sequences play a central role in the representation of functions using infinite series. Sequences are also important because of their usefulness in characterizing a number of important properties of the real numbers. This section, which is devoted to sequences of real numbers and the fundamentals of convergent and divergent sequences, is a prerequisite for Section 7.4.

As defined in Section 4.1, a **sequence** is a function with domain $\mathbb{N}$. If x is a sequence and $n \in \mathbb{N}$, the image of n, usually written x_n instead of $x(n)$, is called the ***n*th term of the sequence *x***.

Examples. The sequence x of odd positive integers has nth term $x_n = 2n - 1$. The first few terms of x are 1, 3, 5, 7, ... The sequence y, where $y_n = (-1)^n$, has range $\{-1, 1\}$ since its terms are alternately -1 and 1.

The sequence x whose nth term is $x_n = \dfrac{(-1)^n}{n+1}$ illustrates the convergence property of sequences. The first few terms of x are $-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$ as shown in Figure 4.6.1. The 99th term is $-\frac{1}{100}$ and the 1000th term is $\frac{1}{1001}$. Evidently

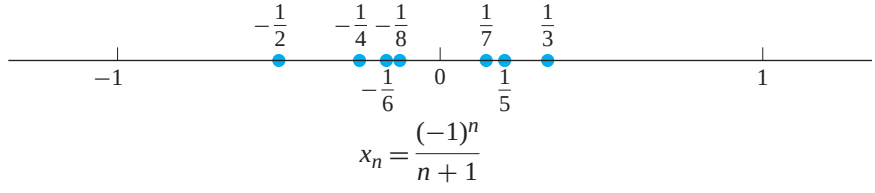


Figure 4.6.1

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