Answers to Selected Exercises

Exercises 1.1

1. **(b)** false
(g) true
2. **(c)** not a proposition; the symbol x acts as a variable
(f) a true proposition
3. **(a)**

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

(c)

P	Q	$\sim Q$	$P \wedge \sim Q$
T	T	F	F
F	T	F	F
T	F	T	T
F	F	T	F

(e)

P	Q	$\sim Q$	$P \wedge Q$	$(P \wedge Q) \vee \sim Q$
T	T	F	T	T
F	T	F	F	F
T	F	T	F	T
F	F	T	F	T

(i)

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$
T	T	T	T	T
F	T	T	T	F
T	F	T	T	T
F	F	T	T	F
T	T	F	T	T
F	T	F	T	F
T	F	F	F	F
F	F	F	F	F

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4. **(a)** false
(c) true
(f) false
(g) true

6. **(a)** equivalent
(c) equivalent
(e) not equivalent
(g) not equivalent

8. **(a)** Since P is equivalent to Q, P has the same truth table as Q. Therefore, Q has the same truth table as P, so Q is equivalent to P.

9. **(c)** tautology

P	Q	$P \wedge Q$	$\sim P \vee \sim Q$	$(P \wedge Q) \vee (\sim P \vee \sim Q)$
T	T	T	F	T
F	T	F	T	T
T	F	F	T	T
F	F	F	T	T

10. **(a)** contradiction
(c) tautology

11. **(a)** x is not positive.
(c) $5 < 3$
(e) Roses are not red or violets are not blue.

13. **(a)** **(i)**

P	Q	$P \ovee Q$
T	T	F
F	T	T
T	F	T
F	F	F

Exercises 1.2

1. **(a)** Antecedent: squares have three sides.
Consequent: triangles have four sides.
(d) Antecedent: f is differentiable.
Consequent: f is continuous.
(f) Antecedent: f is integrable.
Consequent: f is bounded.
(i) Antecedent: An athlete qualifies for the Olympic team.
Consequent: The athlete has a time of 3 minutes, 48 seconds or less.

2. **(a)** Converse: If triangles have four sides, then squares have three sides.
Contrapositive: If triangles do not have four sides, then squares do not have three sides.
(d) Converse: If f is continuous, then f is differentiable.
Contrapositive: If f is not continuous, then f is not differentiable.
(f) Converse: If f is bounded, then f is integrable.
Contrapositive: If f is not bounded, then f is not integrable.

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(j) Converse: A time of 3 minutes, 48 seconds or less is sufficient to qualify for the Olympic team.
Contrapositive: If an athlete records a time that is not 3 minutes and 48 seconds or less, then that athlete does not qualify for the Olympic team.

5. **(a)** true
(c) true
(e) true

6. **(a)** true
(c) true

7. **(b)**

P	Q	$\sim P$	$\sim P \Rightarrow Q$	$Q \Leftrightarrow P$	$(\sim P \Rightarrow Q) \vee (Q \Leftrightarrow P)$
T	T	F	T	T	T
F	T	T	T	F	T
T	F	F	T	F	T
F	F	T	F	T	T

(c)

P	Q	$\sim Q$	$Q \Leftrightarrow P$	$(\sim Q) \Rightarrow (Q \Leftrightarrow P)$
T	T	F	T	T
F	T	F	F	T
T	F	T	F	F
F	F	T	T	T

10. **(a)** f has a relative minimum at $x_0 \wedge f$ is differentiable at $x_0 \Rightarrow f'(x_0) = 0$.
(d) $x = 1 \vee x = -1 \Rightarrow |x| = 1$.
(e) x_0 is a critical point for $f \Leftrightarrow f'(x_0) = 0 \vee f'(x_0)$ does not exist.

11. **(b)** There are three nonequivalent ways to translate the sentence, using the symbols D "The Dolphins make the playoffs" and B "The Bears win all the rest of their games."

$$D \Rightarrow B \text{ or } (\sim B) \Rightarrow (\sim D)$$
$$B \Rightarrow D \text{ or } (\sim D) \Rightarrow (\sim B)$$
$$D \Leftrightarrow B \text{ or } (\sim B) \Leftrightarrow (\sim D)$$

The conditional meaning of *unless* (the first translation) is preferred, but the speaker may have intended any of the three.

12. **(b)**

P	Q	R	$P \wedge Q$	$P \wedge Q \Rightarrow R$	$\sim R$	$\sim Q$	$P \wedge \sim R$	$(P \wedge \sim R) \Rightarrow \sim Q$
T	T	T	T	T	F	F	F	T
F	T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	F	T
F	F	T	F	T	F	T	F	T
T	T	F	T	F	T	F	T	F
F	T	F	F	T	T	F	F	T
T	F	F	F	T	T	T	T	T
F	F	F	F	T	T	T	F	T

Since the fifth and ninth columns are the same, the propositions $P \wedge Q \Rightarrow R$ and $(P \wedge \sim R) \Rightarrow \sim Q$ are equivalent.

13. **(a)** If 6 is an even integer, then 7 is an odd integer.
(c) not possible

16. **(a)** tautology
(d) neither

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Exercises 1.3

1. **(a)** $\sim(\forall x)(x$ is precious $\Rightarrow x$ is beautiful).
Or, $(\exists x)(x$ is precious and x is not beautiful).

(b) *Hint:* This exercise is not the same as 1(a).

(h) $(\forall x)(x \in \mathbb{Z} \Rightarrow x > -4 \vee x < 6)$ or $(\forall x \in \mathbb{Z})(x > -4 \vee x < 6)$

(j) $\sim(\exists x)(\forall y)(x \geq y)$ or $(\forall x)(\exists y)(x < y)$

(l) $(\exists x)(x$ is a positive integer and x is smaller than all other positive integers).
Or, $(\exists x)(x$ is a positive integer and $(\forall y)(y$ is a positive integer $\Rightarrow x \leq y))$.

(m) $(\forall x)(\sim(\forall y)(x$ loves $y))$. Or, $\sim(\exists x)(\forall y)(x$ loves $y)$.

2. **(a)** $(\forall x)(x$ is precious $\Rightarrow x$ is beautiful). All precious stones are beautiful.

(h) $(\exists x \in \mathbb{Z})(x \leq -4 \wedge x < 6)$ There is an integer that is both less than or equal to -4 and greater than or equal to 6.

(j) $(\exists x)(\forall y)(x \geq y)$ There is an integer that is greater than or equal to every integer.

(l) $(\forall x)(x$ is a positive integer $\Rightarrow (\exists y)(y$ is a positive integer$) \wedge x > y))$.
For every positive integer there is a smaller positive integer. Or,
$\sim(\exists x)(x$ is a positive integer $\wedge (\forall y)(y$ is a positive integer $\Rightarrow x \leq y))$.
There is no smallest positive integer.

(m) $(\exists x)(\forall y)(x$ loves $y)$ Someone loves everyone.

5. The first interpretation may be translated as

$$(\forall x)[x \text{ is a person} \Rightarrow (\forall y)(y \text{ is a tax} \Rightarrow x \text{ dislikes } y)].$$

6. **(a)** T, U, V, and W

7. **(b)** *Hint:* Every sentence of the form $P(x)$ is equivalent to $\sim\sim P(x)$. Use this fact to rewrite $(\forall x)(\sim A(x))$ and then simplify by using part (a).

8. **(b)** true

(e) false

(h) true

9. **(b)** Only one real number is both nonnegative and nonpositive.

(d) There is exactly one real number whose natural logarithm is 1.

10. **(a)** true

(d) false

(f) false

(i) false

11. **(a)** *Hint:* Begin by supposing that U is any universe and $A(x)$ is an open sentence.

(b) *Hint:* You must name a specific universe and a specific open sentence such that the converse is false.

(e) $(\forall x)(\sim A(x)) \vee (\exists y)(\exists z)(A(y) \wedge A(z) \wedge y \neq z)$

13. **(d)** This statement is not a denial. It implies the negation of $(\exists! x)P(x)$, but if $(\forall x) \sim P(x)$, then both the statement and $(\exists! x)P(x)$ are false.

14. For every backwards E, there exists an upside down A!

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Exercises 1.4

1. **(a)** Suppose $(G, *)$ is a cyclic group.
$\vdots$
Thus, $(G, *)$ is abelian.
Therefore, if $(G, *)$ is a cyclic group, then $(G, *)$ is abelian.

4. **(a)** The crime took place in the library, not the kitchen. By fact (i), if the crime did not take place in the kitchen, then Professor Plum is guilty. Therefore, Professor Plum is guilty.

5. **(h)** **Proof.** Suppose x is even and y is odd. Then $x = 2k$ for some integer k, and $y = 2j + 1$ for some integer j. Therefore, $x + y = 2k + (2j + 1) = 2(k + j) + 1$, which is odd. (We use the fact that $k + j$ is an integer.)

6. **(d)** *Hint:* The four cases to consider are: case 1, in which $a \geq 0$ and $b \geq 0$; case 2, in which $a < 0$ and $b < 0$; case 3, in which $a \geq 0$ and $b < 0$; and case 4, in which $a < 0$ and $b \geq 0$. In case 3 it is worthwhile to consider two subcases: In subcase (i), $a + b \geq 0$, so that $|a + b| = a + b$; in subcase (ii), $a + b < 0$, so that $|a + b| = -(a + b)$. Now in subcase (i) we have $|a + b| = a + b < a$ (from $b < 0$) and $a < a + (-b)$ (from $0 < -b$). Thus, $|a + b| < a + (-b) = |a| + |b|$. Subcase (ii) is similar. Case 4 is the same as case 3 except for the names of the variables a and b.

7. **(b)** **Proof.** Let a be an integer. Assume that a is even. Then $a = 2k$ for some integer k. Therefore, $a + 1 = 2k + 1$, so $a + 1$ is odd.

(d) **Proof.** *Hint*: Let a be an integer. Use the fact that a is either even or odd to give a proof by cases. It is acceptable, but not necessary, to use the definitions of even and odd in proving these cases: previous exercises have laid the foundations we need. For the case when a is even, use Exercise 7(c) and 5(i). For the case when a is odd, we may use Exercise 5 (e), and then use Exercise 5(i) again.

(g) **Proof.** Suppose a and b are positive integers and a divides b. Then for some integer k, $b = ka$. $\langle$*We must show that $a \leq b$, which is the same as $a \leq ka$. To show $a \leq ka$, we could multiply both sides of $1 \leq k$ by a, using the fact that a is positive. To do that, we must first show $1 \leq k$.*$\rangle$ Since b and a are positive, k must also be positive. Since k is also an integer, $1 \leq k$. Therefore, $a = a \cdot 1 \leq a \cdot k = b$, so $a \leq b$.

(i) **Proof.** Suppose a and b are positive integers and $ab = 1$. Then a divides 1 and b divides 1. By part (g) $a \leq 1$ and $b \leq 1$. But a and b are positive integers, so $a = 1$ and $b = 1$.

10. **(a)** **Proof.** Suppose $A > C > B > 0$. Multiplying by the positive numbers C and B we have $AC > C^2 > BC$ and $BC > B^2$, so $AC > B^2$. AC is positive, so $4AC > AC$. Therefore, $4AC > B^2$, so $B^2 - 4AC < 0$. Thus, the graph must be an ellipse.

11. **(a)** F. This proof, while it appears to have the essence of the correct reasoning, has too many gaps. The first “sentence” is incomplete and the steps

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are not justified. The steps could be justified either by using the definitions or by referring to previous examples and exercises.

(c) C. The order in which the steps are written makes it look as if the author of this "proof" assumed that $x + \dfrac{1}{x} \geq 2$. The proof could be fixed by beginning with the (true) statement that $(x-1)^2 \geq 0$ and ending with the conclusion that $x + \dfrac{1}{x} \geq 2$.

(d) F. This is not a proof of the statement. It is a proof of the converse of the statement.

Exercises 1.5

1. (a) Suppose $(G, *)$ is not abelian.
$\vdots$
Thus $(G, *)$ is not a cyclic group.
Therefore, if $(G, *)$ is a cyclic group, then $(G, *)$ is abelian.

(c) Suppose the set of natural numbers is finite.
$\vdots$
Therefore statement Q.
$\vdots$
Therefore statement $\sim Q$.
This is a contradiction.
Therefore the set of natural numbers is not finite.

(e) **(i)** Suppose that the inverse of the function f from A to B is a function from B to A.
$\vdots$
Therefore f is one-to-one.
$\vdots$
Therefore f is onto B.
Therefore f is one-to-one and onto B.

(ii) Suppose that f is one-to-one and onto B.
$\vdots$
Therefore, the inverse of the function f from A to B is a function from B to A.

3. (a) **Proof.** Suppose that the integer $x + 1$ is not odd. Then $x + 1$ is an even integer. Thus, there exists an integer k such that $x + 1 = 2k$. Then $x = 2k - 1 = 2(k - 1) + 1$, so x is not even. We have shown that if $x + 1$ is not odd, then x is not even. Therefore, if x is even, then $x + 1$ is odd.

(e) *Hint:* The contrapositive statement for "if $x + y$ is even, then either x and y are odd or x and y are even" is "if it is not the case that either x and y are odd or x and y are even then $x + y$ is not even." This is equivalent to "if either x is even and y is odd or x is odd and y is even, then $x + y$ is odd."

4. (b) **Proof.** Suppose it is not true that $2 < x < 3$. Then either $x \leq 2$ or $x \geq 3$. If $x \leq 2$, then $x - 2 \leq 0$ and $x - 3 \leq 0$ $\langle$*since* $x - 3 < x - 2\rangle$.

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Because the product of two nonpositive numbers is nonnegative, $(x-2)(x-3) = x^2 - 5x + 6 \geq 0$. In the other case, if $x \geq 3$, then $x - 3 \geq 0$ and $x - 2 \geq 0$. Therefore $(x-2)(x-3) = x^2 - 5x + 6 \geq 0$. In either case $x^2 - 5x + 6 \geq 0$. We have shown that if $x \leq 2$ or $x \geq 3$, then $x^2 - 5x + 6 \geq 0$. Therefore, if $x^2 - 5x + 6 < 0$ then $2 < x < 3$.

6. (b) **Proof.** Suppose a and b are positive integers. Suppose ab is odd and suppose a and b are not both odd. Then either a is even or b is even. If a is even, then $a = 2k$ for some integer k. Thus $ab = (2k)b = 2(kb)$ is even. Likewise, if b is even, then $b = 2m$ for some integer m and, again, $ab = a(2m) = 2(am)$ is even. Either case leads to a contradiction to the hypothesis that ab is odd. Therefore, if ab is odd then both a and b are odd.

12. (b) A.

Exercises 1.6

1. (a) Choose $m = -3$ and $n = 1$. Then $2m + 7n = 1$.

(c) Suppose m and n are integers and $2m + 4n = 7$. Then 2 divides $2m$ and 2 divides $4n$, so 2 divides their sum $2m + 4n$. But 2 does not divide 7, so this is impossible.

(f) *Hint:* See the statement of part (d). Can you prove that m and n are both negative whenever the antecedent is true?

2. (b) **Proof.** Assume a divides $b - 1$ and $c - 1$. ⟨*The proof involves writing $bc - 1$ as a sum of multiples of a, using the fact that if a divides a number, it divides any multiple of that number* (*Exercise 7(h) of Section* 1.4) *or, more generally, the fact that if a divides two numbers, it divides any sum of multiples of the two numbers (Exercise 2(a)).*⟩ Then a divides the product $(b-1)(c-1) = bc - b - c + 1$. By Exercise 2(a), a divides $(bc - b - c + 1) + (b - 1) = bc - c$. Then also by Exercise 2(a), a divides the sum $(bc - c) + (c - 1) = bc - 1$.

4. (h) *Hint:* For a counterexample, choose $x = 1$. Explain.

(i) *Hint:* For a proof, choose $y = x$. For a different proof, choose $y = 1$.

6. (c) **Proof.** Let n be a natural number. Then both $2n$ and $2n + 1$ are natural numbers. Let $M = 2n + 1$. Then M is a natural number greater than $2n$.

(g) **Proof.** Let $\varepsilon > 0$ be a real number. Then $\frac{1}{\varepsilon}$ is a positive real number and so has a decimal expression as an integer part plus a decimal part. Let M be the integer part of $\frac{1}{\varepsilon}$, plus 1. Then M is an integer and $M > \frac{1}{\varepsilon}$. To prove for all natural numbers $n > M$ that $\frac{1}{n} < \varepsilon$, let n be a natural number and $n > M$. Since $M > \frac{1}{\varepsilon}$, we have $n > \frac{1}{\varepsilon}$. Thus $\frac{1}{n} < \varepsilon$. Therefore, for every real number $\varepsilon > 0$, there is a natural number M such that for all natural numbers $n > M$, $\frac{1}{n} < \varepsilon$.

(h) *Hint:* Because m is positive, the statement $\frac{1}{n} - \frac{1}{m} < \varepsilon$ follows from $\frac{1}{n} < \varepsilon$.

7. (a) F. The false statement referred to is not the opposite (denial) of the claim.

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(b) C. The "proof" shows that there is a polynomial with the required properties, but must also show that there is no other polynomial with these properties.

(d) A.

(i) *Hint*: The grade should be C. What error must be corrected?

Exercises 1.7

1. (a) **Proof.** ⟨*We work both forwards and backwards: From the hypothesis that* $3n + 1$ *is odd we can deduce that* $3n$ *is even, from which we can deduce that* n *is even. We could reach the conclusion that* $2n + 8$ *is divisible by* 4 *if we knew that* 4 *divides* $2n$ *(since* 8 *is divisible by* 4*). In turn, the statement* $2n$ *is divisible by* 4 *may be derived from the statement that* n *is divisible by* 2*. We combine these steps in the proper order to create the proof.*⟩

Suppose n is an integer and $3n + 1$ is odd. Therefore $3n$ is even, which implies that n is even. ⟨*We are now using properties of even and odd integers that we proved earlier, without referencing specific examples or exercises.*⟩ Since n is even, n is divisible by 2. Therefore $2n$ is divisible by 4. Finally since 8 is also divisible by 4, $2n + 8$ is divisible by 4. ■

(b) **Proof.** Let a be a real number, $a \neq 3$. ⟨*The key to the proof is to use the idea of "solution" and then work with the resulting equation.*⟩

Assume that a is a solution to $x^2 - x - 6 = 0$.
Then a makes the equation true ⟨*by the definition of a solution to an equation*⟩.
Thus $a^2 - a - 6 = (a - 3)(a + 2) = 0$.
Then $a + 2 = 0$, because $a - 3 \neq 0$.
Then $(a^2 + 1)(a + 2) = a^3 + 2a^2 + a + 3 = 0$.
Therefore a is a solution to $x^3 + 2x^2 + x + 3 = 0$. ■

(c) **Proof.** Assume that $a \neq 3$. ⟨*Observe that in the proof above, each step implies its predecessor. Thus we can modify the given proof to create an iff proof.*

a is a solution to $x^2 - x - 6 = 0$
iff $a^2 - a - 6 = (a - 3)(a + 2) = 0$.
iff $a + 2 = 0$. ⟨*Because* $a - 3 \neq 0$.⟩
iff $(a^2+1)(a+2) = a^3+2a^2+a+3 = 0$. ⟨*Because* $a^2 + 1 \neq 0$.⟩
iff a is a solution to $x^3 + 2x^2 + x + 3 = 0$. ■

(d) **Proof.** Suppose $x^2 = 2x + 15$ and $x > 2$. Then $(x - 5)(x + 3) = 0$. Since $x > 2$, x must be 5. Then $x - 4$ and $x - 3$ are positive, so $(x - 4)/(x - 3) > 0$. ■

(e) **Proof.** Let x and y be real numbers. ⟨*The statement has the form* $P \Rightarrow (Q \vee R)$*, so it might be proved by assuming* P *and* $\sim Q$ *and deducing* R*. In this case a proof by contrapositive works well.*⟩ Assume that neither x nor y is irrational. Then both x and y are rational, so they can be written in the form $x = \frac{p}{q}$ and $y = \frac{r}{s}$, where p, q, r, and s are integers,

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$q \neq 0$, and $s \neq 0$. Therefore, $x + y = \frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}$. Since $ps + rq$ and qs are integers and $qs \neq 0$, $x + y$ is a rational number. We have shown that if x and y are rational, then $x + y$ is rational. We conclude that if $x + y$ is irrational, then either x or y is irrational. ■

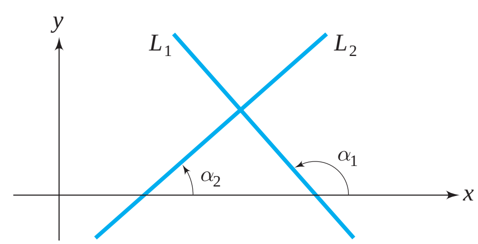
(f) **Proof.** ⟨*If we let S be the set of all nonvertical lines in the xy-plane, we can simplify the symbolic form of the theorem as follows:*

$(\forall L_1 \in S)(\forall L_2 \in S)(L_1$ and L_2 are perpendicular $\Rightarrow$
(slope of L_1) $\cdot$ (slope of L_2) $= -1$).⟩

Let L_1 and L_2 be nonvertical lines. Suppose L_1 and L_2 are perpendicular. ⟨*We now use the fact that the slope of a nonvertical line is tan*(α), *where* α *is the angle of inclination of the line.*⟩ Let α_1 and α_2 be the angles of inclinations of L_1 and L_2, respectively. See the figure. We may assume that $\alpha_1 > \alpha_2$. ⟨*We can make this assumption because the two lines are arbitrary; if* $\alpha_1 < \alpha_2$ *simply interchange the labels of the lines.*⟩ Since L_1 and L_2 are perpendicular, $\alpha_1 = \alpha_2 + \frac{\pi}{2}$. Therefore,

$$\tan(\alpha_1) = \tan(\alpha_2 + \tfrac{\pi}{2}) = -\cot(\alpha_2) = -\frac{1}{\tan(\alpha_2)}.$$

⟨*We use trigonometric identities to rewrite* $\tan(\alpha_1)$.⟩ Thus, $\tan(\alpha_1) \cdot \tan(\alpha_2) = -1$. Since $\tan(\alpha_1)$ is the slope of L_1 and $\tan(\alpha_2)$ is the slope of L_2, the product of the slopes is -1. ■



(g) **Proof.** ⟨*This is a "non-existence" proof. We could restate the result as "Every point inside the circle is not on the line" and begin a direct proof by assuming that (x, y) is a point inside the circle. We would then have to prove that (x, y) is not on the line. In this instance, a better approach is to use a proof by contradiction. The statement has the form*

$$\sim(\exists x)(\exists y)((x,y) \text{ is inside the circle} \wedge (x, y) \text{ is on the line}).\rangle$$

Suppose there is a point (a, b) that is inside the circle and on the line. Then $(a - 3)^2 + b^2 < 6$ and $b = a + 1$. ⟨*We now have two expressions to use.*⟩ Therefore,

$$\begin{aligned} (a-3)^2 + (a+1)^2 &< 6 \\ 2a^2 - 4a + 10 &< 6 \\ a^2 - 2a + 5 &< 3 \\ a^2 - 2a + 1 &< -1 \\ (a-1)^2 &< -1. \end{aligned}$$

This is a contradiction since $(a - 1)^2 \geq 0$. Thus, no point inside the circle is on the line. ■

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(h) **Proof.** ⟨*Proofs that verify equalities or inequalities containing absolute value expressions usually involve cases, because of the two-part definition of* $|x|$. *The two cases are* $x - 2 \geq 0$ *and* $x - 2 < 0$. *The proof in each case is discovered by working backwards from the desired conclusion. The key steps are to note that, in the first case, if* $x \geq 2$, *then* $-6 \leq x$, *and, in the second case, that if* $x \geq 1$, *then* $\frac{6}{7} \leq x$.⟩
Let x be a real number greater than 1.

Case 1. Suppose $x - 2 \geq 0$. Then $|x - 2| = x - 2$. Since $x \geq 2$,

$$-6 \leq x$$
$$3x - 6 \leq 4x$$
$$\frac{3(x-2)}{x} \leq 4. \quad \langle \textit{Remember that } x \textit{ is positive.} \rangle$$

Therefore, $\dfrac{3|x-2|}{x} \leq 4.$

Case 2. Suppose $x - 2 < 0$. Then $|x - 2| = -(x - 2)$. By hypothesis, $x \geq 1$. Therefore,

$$\frac{6}{7} \leq x$$
$$6 \leq 7x$$
$$6 - 3x \leq 4x$$
$$3[-(x-2)] \leq 4x$$
$$\frac{3[-(x-2)]}{x} \leq 4. \quad \langle \textit{Remember that } x \textit{ is positive.} \rangle$$

Therefore, $\dfrac{3|x-2|}{x} \leq 4.$ ■

2. **(e)** *Hint:* Write $n^3 - n$ as the product of 3 consecutive integers.

4. **(b)** *Hint*: Is it possible that for all irrational numbers x and y, $x + y$ is irrational? Or could $x + y = 0$?

8. **(a)** **Proof.** Suppose (x, y) is inside the circle. Then from the distance formula, $(x - 3)^2 + (y - 2)^2 < 4$. Therefore, $|x - 3|^2 < 4$ and $|y - 2|^2 < 4$. It follows that $|x - 3| < 2$ and $|y - 2| < 2$, so $-2 < x - 3 < 2$ and $-2 < y - 2 < 2$. Thus $1 < x < 5$ and $0 < y < 4$. Therefore, $x^2 < 25$ and $y^2 < 16$, so $x^2 + y^2 < 41$.

9. **(d)** $-36 = (-8)5 + 4$. The quotient is -8 and remainder is 4.

14. **(a)** *Hint*: Use Theorem 1.7.3 and then Theorem 1.7.1.

(c) *Hint*: Assume that $d = \gcd(a, b) = 1$ and a divides bc. Write 1 as a linear combination of a and b, and then multiply by c.

16. **(a)** **Proof.** Suppose p is prime and a is any natural number. The only divisors of p are 1 and p, and $\gcd(p, a)$ divides p, so $\gcd(p, a) = 1$ or p. (i) Assume $\gcd(p, a) = p$. Then p divides a by definition of gcd. (ii) Suppose p divides a. Then p is a common divisor of p and a. Since p is the largest divisor of p it is the largest common divisor of p and a, so $\gcd(p, a) = p$.

20. 42

21. **(a)** *Hint*: Use a two-part proof. For the part of the proof that assumes a divides b, show both conditions (i) and (ii) for the lcm are satisfied by b.

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(c) *Hint:* Assume that $\gcd(a, b) = 1$. Since b divides $m = \operatorname{lcm}(a, b)$, $m = kb$ for some integer k. Use part (b) to show that $k \le a$. Then use part (c) of Exercise 14 to show that a divides k. Conclude that $a = k$, so $m = kb = ab$.

(f) *Hint*: By Exercise 14(d), $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$. Use part (c) to find an expression for $\operatorname{lcm}\left(\frac{a}{d}, \frac{b}{d}\right)$. Then use part (e) to obtain another expression for $\operatorname{lcm}\left(\frac{a}{d}, \frac{b}{d}\right)$. Equate the two expressions and simplify.

23. **(b)** A.

Exercises 2.1

1. **(a)** $\{x \in \mathbb{N}: x < 6\}$ or $\{x: x \in \mathbb{N}$ and $x < 6\}$.

(c) $\{x \in \mathbb{R}: 2 \le x \le 6\}$ or $\{x: x \in \mathbb{R}$ and $2 \le x \le 6\}$.

3. **(a)** Suppose that X is a set. If $X \in X$ then X is not an ordinary set, so $X \notin X$. On the other hand, if $X \notin X$, then X is an ordinary set, so $X \in X$. Both $X \in X$ and $X \notin X$ lead to a contradiction. We conclude that the collection of ordinary sets is not a set.

4. **(a)** true
(c) true
(e) false
(g) true
(i) false

5. **(a)** true
(c) true
(e) false
(g) false
(i) false
(k) true

6. **(a)** $A = \{1, 2\}$, $B = \{1, 2, 4\}$, $C = \{1, 2, 5\}$
(c) $A = \{1, 2, 3\}$, $B = \{1, 4\}$, $C = \{1, 2, 3, 5\}$

8. *Hint:* To prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$, begin by assuming that $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$, we recall that $A \subseteq C$ means $(\forall x)(x \in A \Rightarrow x \in C)$. The first steps are: Let x be any object. Suppose $x \in A$. Now use the fact that $A \subseteq B$ and $B \subseteq C$.

9. *Hint:* To prove $A = B$, use the hypothesis $A \subseteq B$ and show $B \subseteq A$ by using Theorem 2.1.1(c).

14. **(a)** $\{\{0\}, \{\triangle\}, \{\square\}, \{0, \triangle\}, \{0, \square\}, \{\triangle, \square\}, X, \varnothing\}$

(c) $\{\{\varnothing\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\varnothing, \{a\}\}, \{\varnothing, \{b\}\}, \{\varnothing, \{a, b\}\}, \{\{a\}, \{b\}\}, \{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}, \{\varnothing, \{a\}, \{b\}\}, \{\varnothing, \{a\}, \{a, b\}\}, \{\varnothing, \{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\}, X, \varnothing\}$.

15. **(a)** false
(e) false
(g) true

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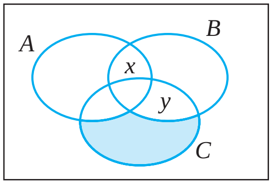
16. **(a)** no proper subsets
(b) $\varnothing$, $\{1\}$, $\{2\}$
17. **(a)** true
(c) true
(e) true
(g) true
(i) true
(k) true
19. **(c)** C. The "proof" asserts that $x \in C$, but fails to justify this assertion with a definite statement that $x \in A$ or that $x \in B$. This problem could be corrected by inserting a second sentence "Suppose $x \in A$" and a fourth sentence "Then $x \in B$."
(e) F. The error repeatedly committed in this proof is to say $A \subseteq B$ means $x \in A$ and $x \in B$. The correct meaning of $A \subseteq B$ is that for every x, if $x \in A$, then $x \in B$.
(h) C. The proof could be considered correct, but it lacks a statement of the hypothesis, helpful explanations and connecting words. How much explanation you include depends on the presumed level of the reader's knowledge. We prefer the use of words, not just symbols.
(i) F. The claim is false. (For example, let $A = \{1, 2\}$, $B = \{1, 2, 4\}$, $C = \{1, 2, 5, 6, 7\}$.) The statement "Since $x \in B$, $x \in A \ldots$," would be correct if we knew that $B \subseteq A$.

Exercises 2.2

1. **(a)** $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
(c) $\{1, 3, 5, 7, 9\}$
(e) $\{3, 9\}$
(g) $\{1, 5, 7\}$
(i) $\{1, 5, 7\}$
2. **(b)** $(1, 8)$
(d) $[2, 4)$
(h) $(-\infty, 3) \cup [8, \infty)$
3. **(a)** $\{0, -2, -4, -6, -8, -10, \ldots\}$
(d) $\mathbb{Z}^- \cup \{0\}$
(g) $\{0, 2, 4, 6, 8, \ldots\}$
4. A and B are disjoint.
6. **(a)** A Venn diagram is helpful.



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Since $C \subseteq A \cup B$, every element of C is either in A or B. In the diagram, the shaded area must be empty. Since $A \cap B$ is not a subset of C, there is some element x that is in $A \cap B$, but not in C. To ensure that C is nonempty, we must place an element y in any one of the three available regions of C. Our solution is $A = \{x\}$, $B = \{x, y\}$, $C = \{y\}$. For other correct examples, we could place other elements anywhere in the diagram (except in the shaded region).

9. **(a)** *Hint:* To prove that $A \subseteq B$ implies $A - B = \varnothing$, assume that $A \subseteq B$ and show that $x \in A - B$ is false for every object x. To prove the converse, assume $A - B = \varnothing$ and there is some object x in A.

(c) **Proof.**

(i) Assume $C \subseteq A \cap B$. Suppose $x \in C$. Then $x \in A \cap B$, so $x \in A$ and $x \in B$. Since $x \in C$ implies $x \in A$, $C \subseteq A$. Similarly, $C \subseteq B$.

(ii) Assume $C \subseteq A$ and $C \subseteq B$. Suppose $x \in C$. Then from $C \subseteq A$ we have $x \in A$ and from $C \subseteq B$ we have $x \in B$. Therefore, $x \in A \cap B$. We conclude that $C \subseteq A \cap B$.

10. **(c)** **Proof.** Suppose $C \subseteq A$ and $D \subseteq B$. Assume that C and D are not disjoint. Then there is an object $x \in C \cap D$. But then $x \in C$ and $x \in D$. Since $C \subseteq A$ and $D \subseteq B$, $x \in A$ and $x \in B$. Therefore, $x \in A \cap B$, so A and B are not disjoint.

11. **(a)** $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3, 4\}$

(c) $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$

(e) $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1\}$

12. **(a)** **Proof.** $S \in \mathcal{P}(A \cap B)$

iff $S \subseteq A \cap B$

iff $\langle$*by Exercise 9(c)*$\rangle$ $S \subseteq A$ and $S \subseteq B$

iff $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$

iff $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

(d) **Proof.** Let A and B be any sets. Since $\varnothing$ is a subset of every set we have $\varnothing \in \mathcal{P}(A - B)$. Also $\varnothing \in \mathcal{P}(A)$ and $\varnothing \in \mathcal{P}(B)$. Therefore $\varnothing \notin \mathcal{P}(A) - \mathcal{P}(B)$. This shows that $\mathcal{P}(A - B) \nsubseteq \mathcal{P}(A) - \mathcal{P}(B)$.

13. **(b)**

$$A \times B = \{(1, q), (1, \{t\}), (1, \pi), (2, q), (2, \{t\}), (2, \pi), (\{1, 2\}, q), (\{1, 2\}, \{t\}), (\{1, 2\}, \pi)\}$$

$$B \times A = \{(q, 1), (\{t\}, 1), (\pi, 1), (q, 2), (\{t\}, 2), (\pi, 2), (q, \{1, 2\}), (\{t\}, \{1, 2\}), (\pi, \{1, 2\})\}$$

15. **(a)** **Proof.** $(a, b) \in A \times (B \cap C)$

iff $a \in A$ and $b \in B \cap C$

iff $a \in A$ and $b \in B$ and $b \in C$

iff $a \in A$ and $b \in B$ and $a \in A$ and $b \in C$

iff $(a, b) \in A \times B$ and $(a, b) \in A \times C$

iff $(a, b) \in (A \times B) \cap (A \times C)$.

19. **(a)** F. One serious error is the assertion that $x \notin C$, which has no justification. The author of this "proof" was misled by supposing $x \in A$, which is an acceptable step but not useful in proving $A - C \subseteq B - C$. After assuming that $A \subseteq B$, the natural first step for proving that $A - C \subseteq B - C$ is to suppose that $x \in A - C$.

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(b) *Hint:* The second sentence of this proof says "Suppose $A - C$," which doesn't make good sense when you think about it. (What should we suppose about the set $A - C$?) Consider that the author of this proof may have meant to say "Suppose $x \in A - C$."

(d) C. The proof that $A \cap B = A$ is incomplete.

(e) F. The claim is false. The statement "$x \in A$ and $x \in \varnothing$ iff $x \in A$" is false.

(i) F. Although a picture may help by suggesting ideas around which a correct proof can be made, a picture alone is rarely sufficient for a proof. Thus a proof that consists only of Venn diagrams will usually have a grade of F. This "proof" is made better because of the explanation that is included, but the only way to give a complete proof is to show that $A \cup B \subseteq B$ and $B \subseteq A \cup B$.

Exercises 2.3

1. **(a)** $\bigcup_{A\in\mathscr{A}} A = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $\bigcap_{A\in\mathscr{A}} A = \{4, 5\}$

(c) $\bigcup_{n\in\mathbb{N}} A_n = \{5, 6, 10, 11, 12, 15, 16, 17, 18\} \cup \{n \in \mathbb{N}\colon n \geq 20\}$; $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$

(e) $\bigcup_{A\in\mathscr{A}} A = \mathbb{Z}$; $\bigcap_{A\in\mathscr{A}} A = \{10\}$

(g) $\bigcup_{n\in\mathbb{N}} A_n = (0, 1)$; $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$

(i) $\bigcup_{r\in\mathbb{R}} A_r = [0, \infty)$; $\bigcap_{r\in\mathbb{R}} A_r = \varnothing$

(p) The union is the triangular region bounded by $y = 0$, $x = 1$, $y = x$. The intersection consists of the sides of the triangle that lie on the axes.

2. The family in Exercise 1(a) is not pairwise disjoint. The family in 1(b) is pairwise disjoint.

4. **(a)** *Hint:* For what real numbers is $(\forall A)(A \in \mathscr{A} \Rightarrow x \in A)$ true?

(b) *Hint:* For what real numbers is $(\exists A)(A \in \mathscr{A} \wedge x \in A)$ true?

5. **(a)** Let $\beta \in \Delta$. Suppose $x \in \bigcap_{\alpha\in\Delta} A_\alpha$. Then $x \in A_\alpha$ for each $\alpha \in \Delta$. Since $\beta \in \Delta$, $x \in A_\beta$. Therefore $\bigcap_{\alpha\in\Delta} A \subseteq A_\beta$.

6. **(a)**
$$\begin{aligned} x \in B \cap \bigcup_{\alpha\in\Delta} A_\alpha \ &\text{iff } x \in B \text{ and } x \in \bigcup_{\alpha\in\Delta} A_\alpha \\ &\text{iff } x \in B \text{ and } x \in A_\alpha \text{ for some } \alpha \in \Delta \\ &\text{iff } x \in B \cap A_\alpha \text{ for some } \alpha \in \Delta \\ &\text{iff } x \in \bigcup_{\alpha\in\Delta} (B \cap A_\alpha). \end{aligned}$$

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7. (a) $$\left(\bigcup_{\alpha\in\Delta} A_\alpha\right) \cap \left(\bigcup_{\beta\in\Gamma} B_\beta\right) = \bigcup_{\beta\in\Gamma}\left(\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)\cap B_\beta\right) = \bigcup_{\beta\in\Gamma}\left(\bigcup_{\alpha\in\Delta}(A_\alpha \cap B_\beta)\right).$$

8. (c) *Hint:* The statement is correct.

9. (a) Suppose $x \in \bigcup_{\alpha\in\Gamma} A_\alpha$. Then $x \in A_\alpha$ for some $\alpha \in \Gamma$. Since $\Gamma \subseteq \Delta$, $\alpha \in \Delta$. Thus $x \in A_\alpha$ for some $\alpha \in \Delta$. Therefore, $x \in \bigcup_{\alpha\in\Delta} A_\alpha$.

10. (a) **Proof.** Let $x \in B$. For each $A \in \mathscr{A}$, $B \subseteq A$. Thus for each $A \in \mathscr{A}$, $x \in A$. Therefore $x \in \bigcap_{A\in\mathscr{A}} A$.

(b) $X = \bigcap_{A\in\mathscr{A}} A$.

15. (c) **Proof.** Let $x \in \bigcup_{i=k}^{m} A_i$. Then there exists $j \in \mathbb{N}$ such that $k \le j \le m$ and $x \in A_j$. Since $j \in \mathbb{N}$ and $x \in A_j$, $x \in \bigcup_{i=1}^{\infty} A_i$.

Alternate Proof: The set $\Gamma = \{k, k+1, k+2, \ldots, m\}$ is a subset of $\Delta = \{1, 2, 3, \ldots\}$. Therefore, by Exercise 9(a), $\bigcup_{i=k}^{m} A_i \subseteq \bigcup_{i=1}^{\infty} A_i$.

17. (a) Let $A_k = \left[-\frac{1}{k}, 1 + \frac{1}{k}\right)$ for each $k \in \mathbb{N}$.

18. (a) C. This proof omits an explanation of why there is some β in Δ such that $A_\beta \in \{A_a : \alpha \in \Delta\}$. The explanation is that by definition of an indexed family, $\Delta \neq \varnothing$. If we allowed $\Delta = \varnothing$, the claim would be false.

(b) C. No connection is made between the first and second sentences. The connection that needs to be made is that if $x \in \bigcup_{\alpha\in\Delta} A_\alpha$ then $x \in A_\alpha$ for some $\alpha \in \Delta$, so $x \in B$ because $A_\alpha \subseteq B$.

(e) F. The claim is false. $\bigcup_{n=1}^{\infty}[n, n+1) = [1, \infty)$.

Exercises 2.4

1. (b) not inductive
(d) not inductive
(f) not inductive

2. (b) true
(e) false

4. (e) $(n+2)(n+1)$

5. (c) $A = \{n: n = 2^k \text{ for some } k \in \mathbb{N}\}$ may be defined as
(i) $2 \in A$.
(ii) if $x \in A$, then $2x \in A$.

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6. **(a)** **Proof.**

(i) The statement is true for $n = 1$ because $\sum_{i=1}^{1}(3i - 2) = 1$ and $\frac{1}{2}(3(1) - 1) = 1$.

(ii) Assume that for some $n \in \mathbb{N}$, $\sum_{i=1}^{n}(3i - 2) = \frac{n}{2}(3n - 1)$. We must show $\sum_{i=1}^{n+1}(3i - 2) = \frac{n+1}{2}(3(n+1) - 1)$. The equation's left-hand side is $\sum_{i=1}^{n}(3i - 2) + [3(n+1) - 2] = \frac{n}{2}(3n - 1) + (3n + 1) = \frac{3}{2}n^2 + \frac{5}{2}n + 1$ and the equation's right-hand side is $\frac{1}{2}(n+1)(3n+2)$, which also simplifies to $\frac{3}{2}n^2 + \frac{5}{2}n + 1$.
Thus the statement is true for $n + 1$.

(iii) By the PMI, the statement is true for every $n \in \mathbb{N}$.

7. **(b)** **Proof.**

(i) For $n = 1$, $4^1 - 1 = 3$, which is divisible by 3.

(ii) Suppose for some $k \in \mathbb{N}$ that $4^k - 1$ is divisible by 3. Then

$$\begin{aligned} 4^{k+1} - 1 &= 4(4^k) - 1 \\ &= 4(4^k - 1) - 1 + 4 \\ &= 4(4^k - 1) + 3. \end{aligned}$$

Both 3 and $4^k - 1$ are divisible by 3, so $4^{k+1} - 1$ is divisible by 3.

(iii) By the PMI, $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

(i) **Proof.**

(i) $3^{3+1} = 3^4 = 81 > 64 = (1 + 3)^3$, so the statement is true for $n = 1$.

(ii) Assume that $3^{n+3} > (n+3)^3$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned} 3^{(n+1)+3} &= 3^{n+4} = 3 \cdot 3^{n+3} \\ &> 3(n+3)^3 = 3(n^3 + 9n^2 + 27n + 27) \\ &= 3n^3 + 27n^2 + 81n + 81 \\ &> n^3 + 12n^2 + 48n + 64 = (n+4)^3 = ((n+1)+3)^3. \end{aligned}$$

That is, $3^{(n+1)+3} > ((n+1)+3)^3$.

(iii) By the PMI, $3^{n+3} > (n+3)^3$ for every $n \in \mathbb{N}$.

8. **(a)** **Proof.**

(i) $6^3 = 216 < 720 = 6!$, so the statement is true for $n = 6$.

(ii) Assume $n^3 < n!$ for some $n \geq 6$. Then

$$\begin{aligned} (n+1)^3 &= n^3 + 3n^2 + 3n + 1 \\ &< n^3 + 3n^2 + 3n + n = n^3 + 3n^2 + 4n \\ &< n^3 + 3n^2 + n^2 = n^3 + 4n^2 \\ &< n^3 + n^3 = 2n^3 \\ &< (n+1)n^3 \\ &< (n+1)n! = (n+1)!. \end{aligned}$$

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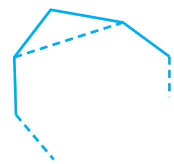
(iii) By the PMI, $n^3 < n!$ for all $n \geq 6$.

(c) *Hint:* $(n+2)[(n+1)!] > (n+2)2^{n+3} > 2(2^{n+3})$.

(g) *Hint:* For a convex polygon, any line segment drawn from an edge point to another edge point lies inside the polygon. For the inductive step, when you consider a polygon of $n+1$ sides, draw a line as shown between two vertices. (Only part of the polygon is shown.)



The new line segment separates the upper left triangle from a convex polygon that has exactly n sides, so we can apply the hypothesis of induction to the n-sided polygon. Then use the result to compute the sum of the interior angles for the polygon of $n+1$ sides.

10. *Hint:* Use $n = 3$ for the basis step. For the inductive step, assume the statement is correct for any collection of n points with no three points collinear. Consider a collection of $n+1$ points, but apply the hypothesis of induction to only n of those points. Then calculate the total number of line segments determined by all $n+1$ points.

11. *Hint:* For the induction step, visualize the starting position with $n+1$ disks. When you think about the moves you would make to transfer all $n+1$ disks from one peg to another, try to break down the task into three separate tasks, so you can use the assumption about how many moves are required to move n disks. The first task is to move the top n disks from the stack to another peg.

13. **(a)** F. The claim is obviously false, but this example of incorrect reasoning is well known because it's fun and the flaw is not easy to spot.

Let $S = \{n \in \mathbb{N}$: all horses in every set of n horses have the same color$\}$. It is true that $1 \in S$. It is also true, **for $n \geq 2$**, that if $n \in S$, then $n+1 \in S$. The "proof" fails in the case when $n = 1$ because $1 \in S$ but $2 \notin S$. In this case, when either horse is removed from the set, the remaining horse has the same color (as itself), [because there is only one horse left] but the two horses may have different colors. We conclude that the set S is not inductive, and in fact $S = \{1\}$.

(b) F. The basis step and the assumption that the statement is true for some n are correct. Perhaps the author hopes that just saying the statement is true for $n+1$ is good enough. For a correct proof, one must use the statement about n to prove the statement about $n+1$.

(e) F. The factorization of $xy+1$ is wrong, and there is no reason to believe $x+1$ or $y+1$ is prime.

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Exercises 2.5

1. **(a)** *Hint*: Let $S = \{n \in \mathbb{N}\colon n > 22 \text{ and } n = 3s + 4t \text{ for some integers } s \geq 3 \text{ and } t \geq 2\}$. Show that 23, 24, and 25 are in S. Then let $m > 22$ be a natural number and assume that for all $k \in \{23, 24, \ldots m - 1\}$, $k \in S$. To show that $m \in S$, proceed as follows: If $m = 23$, 24, or 25 we already know m is in S. Otherwise $m \geq 26$, so $m - 3 \geq 23$. By the hypothesis of induction, $m - 3 \in S$, so $m - 3 = 3s + 4t$ for some integers s and t, where $s \geq 3$ and $t \geq 2$. Then $m = 3(s + 1) + 4t$.

3. **(c)** *Hint:* Let $A = \{b \in \mathbb{N}\colon \text{there exists } a \in \mathbb{N} \text{ such that } a^2 = 2b^2\}$, assume A is nonempty, and use the WOP to reach a contradiction.

5. **(a)** *Hint:* The induction hypothesis is "Suppose f_{3k} is even and both f_{3k+1} and f_{3k+2} are odd for some natural number k." From this use the definition of Fibonacci numbers to show that $f_{3(k+1)}$ is even and both $f_{3(k+1)+1}$ and $f_{3(k+1)+2}$ are odd.

(d) **Proof.**

(i) In the case of $n = 1$, the formula is $f_1 = f_{1+2} - 1$, which is $1 = 2 - 1$. Thus the statement is true for $n = 1$.

(ii) Suppose for some k that $f_1 + f_2 + f_3 + \cdots + f_k = f_{k+2} - 1$. Then

$$\begin{aligned} f_1 + f_2 + f_3 + \cdots + f_k + f_{k+1} &= (f_1 + f_2 + f_3 + \cdots + f_k) + f_{k+1} \\ &= (f_{k+2} - 1) + f_{k+1} \\ &= (f_{k+2} + f_{k+1}) - 1 \\ &= f_{k+3} - 1. \end{aligned}$$

Therefore the statement is true for $n + 1$.

(iii) Thus, by the PMI, $f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$ for all natural numbers n.

6. **(d)** *Hint*: Consider the cases $n = 1$ and $n = 2$ separately. For $n > 2$, you will find it useful to multiply the equation $\alpha^2 = \alpha + 1$ by α^{n-2} and $\beta^2 = \beta + 1$ by β^{n-2}.

7. *Hint*: Modify the proof given for the case $a > 0$. Slightly different arguments are needed to show (i) that S is nonempty and (ii) that $r < |a| = -a$.

9. *Hint*: Suppose n is the smallest positive integer greater than 1 that is not prime and such that n can be expressed in two different ways as a product of primes (where we ignore the order in which the prime factors appear). Then n may be written as: $n = p_1 p_2 p_3 \ldots p_n$ and $n = q_1 q_2 q_3 \ldots q_m$. Apply Euclid's Lemma to show that $p_1 = q_j$ for some $1 \leq j \leq m$. Then find a contradiction.

12. **(b)** **Proof.** Let S be a subset of $\mathbb{N}$ such that $1 \in S$ and S is inductive. We wish to show that $S = \mathbb{N}$. Assume that $S \neq \mathbb{N}$ and let $T = \mathbb{N} - S$. By the WOP, the nonempty set T has a least element. This least element is not 1, because $1 \in S$. If the least element is n, then $n \in T$ and $n - 1 \in S$. But by the inductive property of S, $n - 1 \in S$ implies that $n \in S$. This is a contradiction. Therefore, $S = \mathbb{N}$.

13. **(b)** F. The claim is false. The flaw in the "proof" is the incorrect assumption that $m - 1$ is a natural number. In fact, 1 *is* the smallest natural number n

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such that 3 does not divide $n^3 + 2n + 1$. There is no contradiction about a smaller natural number because there is no smaller natural number.

Exercises 2.6

2. **(b)** 16

3. *Hint:* Since $1{,}000{,}000 = (10^3)^2 = (10^2)^3 = 10^6$, there are 10^3 squares less than or equal to 1,000,000; 10^2 cubes less than or equal to 1,000,000; and 10 natural numbers that are both squares and cubes (sixth powers) less than or equal to 1,000,000.

5. *Hint*: Complete this formula:

$$\overline{\overline{A \cup B \cup C \cup D}} = \overline{\overline{A}} + \cdots - \overline{\overline{A \cap B}} - \cdots + \overline{\overline{A \cap B \cap C}} + \cdots - \overline{\overline{A \cap B \cap C \cap D}}.$$

10. *Hint*: The answer is not $20 \cdot 19 \cdot 19 \cdot 18$; it is 130,340. Consider the cases where the bottom right is colored the same as or differently from the upper left corner, and give two products that yield the correct sum.

21. **(a)** *Hint*: The algebra in the inductive step is:

$$\begin{aligned}
&\sum_{r=0}^{n+1}\binom{n+1}{r}a^r b^{n+1-r}\\
&= b^{n+1} + \sum_{r=1}^{n}\binom{n+1}{r}a^r b^{n+1-r} + a^{n+1}\\
&= b^{n+1} + \sum_{r=1}^{n}\left(\binom{n}{r} + \binom{n}{r-1}\right)a^r b^{n+1-r} + a^{n+1}\\
&= b^{n+1} + \sum_{r=1}^{n}\binom{n}{r}a^r b^{n+1-r} + \sum_{r=1}^{n}\binom{n}{r-1}a^r b^{n+1-r} + a^{n+1}\\
&= b\left(b^n + \sum_{r=1}^{n}\binom{n}{r}a^r b^{n-r}\right) + a\left(\sum_{r=1}^{n}\binom{n}{r-1}a^{r-1}b^{n+r-1} + a^n\right)\\
&= b\left(\sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}\right) + a\left(\sum_{r=0}^{n-1}\binom{n}{r}a^r b^{n-r} + a^n\right)\\
&= b\left(\sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}\right) + a\left(\sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}\right)\\
&= (a+b)\left(\sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}\right).
\end{aligned}$$

(c) **Proof.** Choose one particular element x from a set A of n elements. The number of subsets of A with r elements is $\binom{n}{r}$. The collection of r-element subsets may be divided into two disjoint collections: those subsets containing x and those subsets not containing x. We count the number of subsets in each collection and add the results. First, there are $\binom{n-1}{r}$ r-element subsets of A that do not contain x, since each is a subset of $A - \{x\}$. Second, there are $\binom{n-1}{r-1}$ r-element subsets of A that do

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contain x, because each of these corresponds to the $(r-1)$-element subset of $A - \{x\}$ obtained by removing x from the subset. Thus the sum of the number of subsets in the two collections is

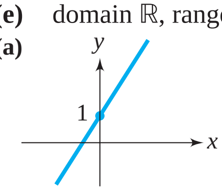
$$\binom{n-1}{r} + \binom{n-1}{r-1} = \binom{n}{r}.$$

23. **(b)** *Hint:* Consider two disjoint sets containing n and m elements.

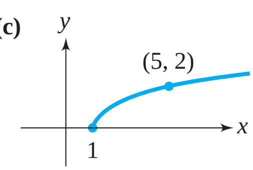
Exercises 3.1

2. **(a)** domain $\mathbb{R}$, range $\mathbb{R}$
(c) domain $[1, \infty)$, range $[0, \infty)$
(e) domain $\mathbb{R}$, range $\mathbb{R}$

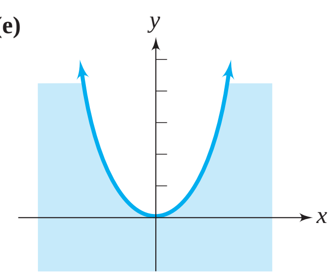
3. **(a)**



(c)



(e)



4. **(a)** $R_1^{-1} = R_1$
(c) $R_3^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{1}{7}(x + 10)\right\}$
(e) $R_5^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \pm\sqrt{\frac{5 - x}{4}}\right\}$
(g) $R_7^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : y > \frac{x + 4}{3}\right\}$
(i) $R_9^{-1} = \{(x, y) \in P \times P$: y is a child of x and x is male$\}$

5. **(b)** $R \circ T = \{(3, 2), (4, 5)\}$
(d) $R \circ R = \{(1, 2), (2, 2), (5, 2)\}$

6. **(a)** $R_1 \circ R_1 = \{(x, y) : x = y\} = R_1$
(d) $R_2 \circ R_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = -35x + 52\}$
(g) $R_4 \circ R_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 16x^4 - 40x^2 + 27\}$

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(j) $R_6 \circ R_6 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y < x + 2\}$

(n) $R_3 \circ R_8 = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \dfrac{14x}{x-2} - 10\right\}$

(p) *Hint:* $R_9 \circ R_9$ is *not* $\{(x, y): y$ is a grandfather of $x\}$.

15. **(a)** F. The statements "$x \in A \times B$" and "$x \in A$ and $x \in B$" are not equivalent. It's wise to avoid writing something like "Suppose $x \in A \times B$." Think of an element of a product as an ordered pair, and write "Suppose $(x, y) \in A \times B$."

(b) C. The only correction required is that $(a, c) \notin B \times D$ implies $a \notin B$ *or* $c \notin D$.

(d) A.

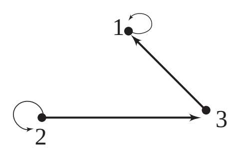
Exercises 3.2

1. **(a)** not reflexive, not symmetric, transitive

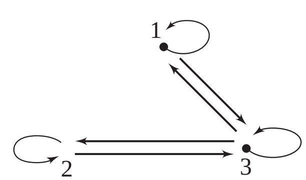
(e) reflexive, not symmetric, transitive

(l) not reflexive, symmetric, not transitive (*Note:* Sibling means "a brother or sister.")

2. **(a)** $\{(1, 1), (2, 2), (2, 3), (3, 1)\}$

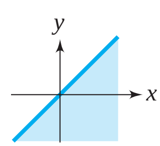


(d) $\{(1, 1), (2, 2), (3, 3), (1, 3), (2, 3), (3, 1), (3, 2)\}$



3. **(a)** Sketch the graph $y = 2x$. This relation is not reflexive on $\mathbb{R}$ because it does not contain $(1, 1)$, not symmetric because it contains $(1, 2)$ but not $(2, 1)$, and not transitive because it contains $(1, 2)$ and $(2, 4)$ but not $(1, 4)$.

(d) *Hint:* Sketch the line $y = x$ and the unit circle. This relation is not transitive because it contains $(1, 0)$ and $(0, -1)$ but not $(1, -1)$.

(f) 

This is the graph of the relation $\{(x, y): y \le x\}$.

5. **(d)** *Hint:* To show that R is reflexive, let a be a natural number. All prime factorizations of a have the same number of 2's. Thus $a\ R\ a$. It must also

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be shown that R is symmetric and transitive. Three elements of $4/R$ are $4 = 2 \cdot 2$, $28 = 2 \cdot 2 \cdot 7$, and $300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$.

(g) *Hint*: First show P is reflexive on $\mathbb{R} \times \mathbb{R}$ and symmetric. To show transitivity begin by supposing that $(x, y)\ P\ (z, w)$ and $(z, w)\ P\ (u, v)$. Then $|x - y| = |z - w|$ and $|z - w| = |u - v|$.

The equivalence class of $(0, 0)$ is the line $y = x$.

7. **(a)** transitive, but not reflexive and not symmetric

(c) reflexive, symmetric, and transitive

8. **(a)** $\overline{0} = \{\ldots, -15, -10, -5, 0, 5, 10, \ldots\}$
$\overline{1} = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$
$\overline{2} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$
$\overline{3} = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$
$\overline{4} = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$

10. **(b)** *Hint:* See part (a).

(c) *Hint:* Assume $x \equiv_m y$. Part 1: Suppose $z \in \bar{x}$. Then $x \equiv_m z$. By symmetry $y \equiv_m x$ and by transitivity $y \equiv_m z$. Thus $z \in \bar{y}$. This shows $\bar{x} \subseteq \bar{y}$. In part 2, we must show $\bar{y} \subseteq \bar{x}$.

13. **(b)** ***Proof.*** Assume R is symmetric. Then $(x, y) \in R$ iff $(y, x) \in R$ iff $(x, y) \in R^{-1}$. Thus $R = R^{-1}$. Now, suppose $R = R^{-1}$. Then $(x, y) \in R$ implies $(x, y) \in R^{-1}$, which implies $(y, x) \in R$. Thus R is symmetric.

15. **(a)** ***Proof.*** Suppose $(x, y) \in R \cup R^{-1}$. Then $(x, y) \in R$ or $(x, y) \in R^{-1}$. If $(x, y) \in R$, then $(y, x) \in R^{-1}$. Likewise, if $(x, y) \in R^{-1}$, then $(y, x) \in R$. In either case, $(y, x) \in R \cup R^{-1}$. Thus, $R \cup R^{-1}$ is symmetric.

18. *Hint*: One part of the proof is to show that R is symmetric. Suppose $x\ R\ y$. Then $x\ L\ y$ and $y\ L\ x$, so $y\ L\ x$ and $x\ L\ y$ Therefore, $y\ R\ x$.

19. **(f)** F. The last sentence confuses $R \cap S$ with $R \circ S$. A correct proof requires a more complete second sentence.

Exercises 3.3

2. **(d)** The elements of $\mathscr{A}$ are natural numbers, not subsets of $\mathbb{N}$, so $\mathscr{A}$ is not a partition of $\mathbb{N}$. Note: $\{\{1, 2, 3, 4\}, \{n \in \mathbb{N}: n > 5\}\}$ *is* a partition of $\mathbb{N}$.

3. **(b)** There are 10 equivalence classes. The class $0/R$ contains $0, 1, 2, \ldots, 9, 100, 101, \ldots 109, 200, 201, \ldots, 209, \ldots$ and all the negatives of these numbers. The class of 10 modulo R contains all integers that have 1 as the tens digit, and so forth.

5. *Hint:* There are four subsets in the partition. One of them is the set $\{(1,-1), (-1, 1), (i, i), (-i, -i)\}$.

6. **(b)** $x\ R\ y$ iff $x = y$ or both $x > 2$ and $y > 2$.

(d) $x\ R\ y$ iff (i) $x = y$ and $x \in \mathbb{Z}$ or (ii) $\text{int}(x) = \text{int}(y)$ and $x, y \notin \mathbb{Z}$. (Recall that $\text{int}(x)$ denotes the greatest integer function.)

8. **(a)** $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5,5)\}$

11. No. Let R be the relation $\{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1)\}$ on the set $A = \{1, 2, 3\}$. Then $R(1) = \{1, 2, 3\}$, $R_2 = \{1, 2\}$, and $R(3) = \{1, 3\}$. The set $\mathscr{A} = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}\}$ is not a partition of A.

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14. **(a)** Yes $\{B_1^c, B_2^c\}$ of a partition of A when $B_1 \neq B_2$, because $\{B_1^c, B_2^c\} = \{B_2, B_1\}$. If $B_1 = B_2$, then $B_1 = B_2 = A$ and $B_1^c = B_2^c = \varnothing$, so $\{B_1^c, B_2^c\}$ is not a partition of A.

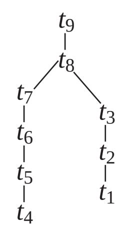
15. **(d)** A (or C). The proof is correct because the ideas are all there, and every statement is true. You may give it a C if you feel the ideas are not well connected.

Exercises 3.4

1. **(a)** No, since (2, 4) and (4, 2) are in R.
(c) No, since (2, −2) and (−2, 2) are in R.
(f) No, since (1, 3) and (3, 1) are in R.

10. **(a)** $\{(a, a), (b, b), (c, c), (c, a), (c, b)\}$

11. **(a)** There are multiple correct answers for this question, depending on preferences. One answer is:



12. **(a)** **Proof.** Suppose $B - \{x\} \subseteq C \subseteq B$. For any $y \in B - \{x\}$, we have $y \in C$. If $x \in C$, then for all $y \in B$ we have $y \in C$. Therefore, $B \subseteq C$, which shows that $C = B$. On the other hand, if $x \notin C$, then for all $y \in C$ we have $y \in B$ and $y \neq x$. Thus, $C \subseteq B - \{x\}$, which shows that $C = B - \{x\}$. Therefore, there is no C different from B and $B - \{x\}$ such that $B - \{x\} \subseteq C \subseteq B$. Hence $B - \{x\}$ is an immediate predecessor of B.

13. **(b)** No. For example, consider a set of two squares where the squares are side by side within the rectangle.
(c) A set containing two disjoint squares does not have a lower bound.

14. **(a)** *Hint:* Use parts of Theorem 2.2.1 and Exercise 9 of Section 2.2.
(b) *Hint:* Use parts of Theorem 2.3.1 and Exercise 10 of Section 2.3.

20. **(b)** F. This proof does not show that $\sup(B)$ exists. All it shows is that *if* $\sup(B)$ *exists*, then $u = \sup(B)$. A correct proof would show that $u = \sup(B)$ by showing u has the two supremum properties (u is an upper bound and $u \, R \, v$ for all other upper bounds v.)

Exercises 3.5

2. **(d)** not possible

3. **(c)** not possible

9. *Hint:* Suppose the graph G has order $n \geq 2$, and all vertices have different degrees. Consider what these degrees must be. Can one vertex have degree $n - 1$ and another have degree 0?

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Exercises 4.1

1. **(a)** This is a function with domain and range $\{0, \triangle, \square, \cup, \cap\}$. Other possible codomains are $\{\triangle, \square, \cup, \cap, \$\}$, $\{0, \triangle, \square, \cup, \cap, +, \#, \%\}$ and $\{\triangle, \square, \cup, \cap, 0, 1, 2, 3\}$.

3. **(a)** Domain $= \mathbb{R} - \{-1\}$. Range $= \{y \in \mathbb{R}: y \neq 0\}$. A possible codomain is $\mathbb{C}$.

(d) Domain $= \mathbb{R} - \left\{\frac{\pi}{2} + k\pi: k \in \mathbb{Z}\right\}$. Range $= \mathbb{R}$. A possible codomain is $\mathbb{C}$.

4. **(a)** $\text{Dom}(f) = \mathbb{R} - \{3\}$, $\text{Rng}(f) = \mathbb{R} - \{-1\}$.

5. **(a)** If $x = 1$, then $2x + y = 2 + y$. For $y = 1, 2, 3$, or 4, $2 + y$ is prime and not 5 iff $y = 1$.
If $x = 2$ then $2x + y = 4 + y$. For $y = 1, 2, 3$, or 4, $4 + y$ is prime and not 5 iff $y = 3$.
If $x = 3$ then $2x + y = 6 + y$. For $y = 1, 2, 3$, or 4, $6 + y$ is prime and not 5 iff $y = 1$.
If $x = 4$ then $2x + y = 8 + y$. For $y = 1, 2, 3$, or 4, $8 + y$ is prime and not 5 iff $y = 3$.
For each $x \in A$, there is a unique y in A such that $(x, y) \in R$.

8. **(a)** A.

9. **(a)** Let $x_n = -n$. Then x is the sequence $-1, -2, -3, -4, \ldots$.

(c) *Hint:* Find a rule so that $x_1 = 3\frac{1}{2}$ and $x_2 = 3\frac{2}{3}$.

10. **(a)** $f(3) = \overline{3} = \{\ldots, -9, -3, 3, 9, \ldots\}$.

11. **(b)** This rule is a function.

(e) Not a function. For example in $\mathbb{Z}_3$ we have $\overline{0} = \overline{3} = \overline{6} = \overline{9}$. The rule assigns to $\overline{0}$ these different images: [0], [3], [2], and [1] in $\mathbb{Z}_4$.

15. **(a)** $\text{Dom}(S)$

16. **(a)** ***Proof.*** Let $x, y, z \in \mathbb{N}$.

(i) By definition of absolute value, $d(x, y) = |x - y| \geq 0$ for all $x, y \in \mathbb{N}$.
(ii) $d(x, y) = |x - y| = 0$ iff $x - y = 0$ iff $x = y$.
(iii) $d(x, y) = |x - y| = |y - x| = d(y, x)$.
(iv) By the triangle property of absolute value, $|x - y| + |y - z| \geq |x - z|$. Thus $d(x, y) + d(y, z) \geq d(x, z)$.

17. **(c)** $\binom{m}{2}n^2$ We choose 2 elements from A for first coordinates; each may then be assigned any element of B as its image.

Exercises 4.2

1. **(a)** $(f \circ g)(x) = 17 - 14x$, $(g \circ f)(x) = -29 - 14x$

(c) $(f \circ g)(x) = \sin(2x^2 + 1)$, $(g \circ f)(x) = 2\sin^2 x + 1$

(e) $f \circ g = \{(k, r), (t, r), (s, l)\}$, $g \circ f = \varnothing$

(i) Observe that $(f \circ g)(x) = \begin{cases} f(2x) & \text{if } x \leq -1 \\ f(-x) & \text{if } x > -1 \end{cases}$.

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We must consider cases. If $x \le -1$, $f(g(x)) = f(2x)$. Since in this case $2x \le 0$, $f(2x) = 2x + 1$. If $x > -1$, $f(g(x)) = f(-x)$. There are two sub-cases:

If $x \ge 0$, then $-x \le 0$, so $f(-x) = -x + 1$.
If $-1 < x < 0$, then $1 > -x > 0$, so $f(-x) = -2x$.

Therefore,

$$(f \circ g)(x) = \begin{cases} 2x+1 & \text{if } x \le -1 \\ -2x & \text{if } -1 < x < 0. \\ -x+1 & \text{if } x \ge 0 \end{cases}$$

Similarly,

$$(g \circ f)(x) = \begin{cases} g(x+1) & \text{if } x \le 0 \\ g(2x) & \text{if } x > 0 \end{cases}.$$

If $x \le 0$, $g(f(x)) = g(x+1)$. There are two subcases:

If $x \le -2$, then $x + 1 \le -1$, so $g(x+1) = 2x + 2$.
If $x > -2$, then $x + 1 > -1$, so $g(x+1) = -x - 1$.

If $x > 0$, $g(f(x)) = g(2x)$. Since $x > 0$, $2x > 0$, so $2x > -1$. Thus $g(2x) = -2x$.

Therefore, $(g \circ f)(x) = \begin{cases} 2x+2 & \text{if } x \le -2 \\ -x-1 & \text{if } -2 < x \le 0. \\ -2x & \text{if} x > 0 \end{cases}$

2. **(a)** $\text{Dom}(f \circ g) = \mathbb{R} = \text{Rng}(f \circ g) = \text{Dom}(g \circ f) = \text{Rng}(g \circ f)$
(c) $\text{Dom}(f \circ g) = \mathbb{R}$, $\text{Rng}(f \circ g) = [-1, 1]$
$\text{Dom}(g \circ f) = \mathbb{R}$, $\text{Rng}(g \circ f) = [1, 3]$
(e) $\text{Dom}(f \circ g) = \{k, t, s\}$, $\text{Rng}(f \circ g) = \{r, l\}$
$\text{Dom}(g \circ f) = \varnothing = \text{Rng}(g \circ f)$
(i) $\text{Dom}(f \circ g) = \mathbb{R}$, $\text{Rng}(f \circ g) = (-\infty, 2)$
$\text{Dom}(g \circ f) = \mathbb{R}$, $\text{Rng}(g \circ f) = (-\infty, 1)$

3. **(a)** Example 1: $f(x) = x^2$, $g(x) = 3x + 7$.
Example 2: $f(x) = (x+7)^2$, $g(x) = 3x$.

5. **(a)** $f^{-1}(x) = \dfrac{x-2}{5}$
(c) $f^{-1}(x) = \dfrac{1-2x}{x-1}$
(e) $f^{-1}(x) = -3 + \ln x$

9. **(a)** $\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = 0 \text{ if } x < 0 \text{ and } y = x^2 \text{ if } x \ge 0\}$
$\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x^2\}$

13. *Hint:* Write $A \cup C$ as $A \cup (C - E)$. Then show $h \cup g = h \cup (g|_{C-E})$ and use Theorem 4.2.5.

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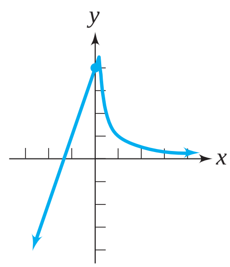
14. **(a)** $h \cup g$ is a function.



16. **(a)** ***Proof.*** Let $x, y \in \mathbb{R}$. Suppose. $x < y$. Then $3x < 3y$ and $3x - 7 < 3y - 7$. There-fore, $f(x) < f(y)$.

(d) ***Proof.*** Suppose x and y are in $(-3, \infty)$ and $x < y$. Then $4x < 4y$, so $3x - y < 3y - x$. Thus $xy + 3x - y - 3 < xy + 3y - x - 3$ or $(x-1)(y+3) < (y-1)(x+3)$. Using the fact that x and y are in $(-3, \infty)$, we know $x + 3$ and $y + 3$ are positive. Dividing both sides of the last inequality by $x + 3$ and $y + 3$, we conclude that

$$\frac{x-1}{x+3} < \frac{y-1}{y+3}.$$

Thus $f(x) < f(y)$.
⟨*Note: This proof was found by working backward from the conclusion.*⟩

17. **(d)** The function f given by

$$f(x) = \begin{cases} x+1 & \text{if } x \le 0 \\ 1-x & \text{if } 0 < x < 1 \\ x-1 & \text{if } x \ge 1 \end{cases}$$

is a counterexample. Another counterexample is f given by $f(x) = (x+1)(x-1)^2$.

18. **(a)** ***Proof.*** We show that $f_1 + f_2$ is a function with a domain $\mathbb{R}$. First $f_1 + f_2$ is by definition a relation. For all $x \in \mathbb{R}$ there is some $u \in \mathbb{R}$ such that $(x, u) \in f_1$ because $f_1\colon \mathbb{R} \to \mathbb{R}$ and there exists $v \in \mathbb{R}$ such that $(x, v) \in f_2$ because $f_2\colon \mathbb{R} \to \mathbb{R}$. Then $(x, u + v) \in f_1 + f_2$, so $x \in \text{Dom}(f_1 + f_2)$. It is clear from the definition of $f_1 + f_2$ that $x \in \text{Dom}(f_1 + f_2)$ implies $x \in \mathbb{R}$, so $\text{Dom}(f_1 + f_2) = \mathbb{R}$.

Let $x \in \mathbb{R}$. Suppose (x, c) and (x, d) are in $f_1 + f_2$. Then $c = f_1(x) + f_2(x) = d$. Therefore, $f_1 + f_2$ is a function.

20. **(a)** A. On line 3 the author of this proof chose not to mention that $x \in A$ (because A is the domain of f). The reader is expected to observe this to verify that $(x, y) \in I_A$.

Exercises 4.3

1. **(a)** Onto $\mathbb{R}$. ***Proof.*** Let $w \in \mathbb{R}$. Then for $x = 2(w - 6)$, $x \in \mathbb{R}$, and $f(x) = \frac{1}{2}[2(w-6)] + 6 = w$. Thus $w \in \text{Rng}(f)$. Therefore, f maps onto $\mathbb{R}$.

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(c) Not onto $\mathbb{N} \times \mathbb{N}$. Since $(5, 8) \in \mathbb{N} \times \mathbb{N}$ and $(5, 8) \notin \text{Rng}(f)$, f does not map onto $\mathbb{N} \times \mathbb{N}$.

(k) ***Proof.*** First, if $x \in [2, 3)$, then $x - 2 \geq 0$ and $3 - x > 0$, so $f(x) = \dfrac{x-2}{3-x} \geq 0$. Therefore $\text{Rng}(f) \subseteq [0, \infty)$. Now let $w \in [0, \infty)$. Choose $x = \dfrac{3w+2}{w+1}$. Then $w \geq 0$, so $2w + 2 \leq 3w + 2 < 3w + 3$. Dividing by $w + 1$, we have $2 \leq x < 3$. Thus $x \in [2, 3)$, and

$$f(x) = \left[\frac{3w+2}{w+1} - 2\right] \div \left[3 - \frac{3w+2}{w+1}\right]$$
$$= \frac{3w + 2 - 2(w+1)}{3(w+1) - (3w+2)} = w.$$

⟨*This value for x was found by working backward from the desired result.*⟩ Therefore, f maps onto $[0, \infty)$.

2. (a) One-to-one. ***Proof.*** Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{R}$. Then $\frac{1}{2}x + 6 = \frac{1}{2}y + 6$. Then $\frac{1}{2}x = \frac{1}{2}y$, so $x = y$.

(c) One-to-one. ***Proof.*** Suppose $m, n \in \mathbb{N}$ and $f(m) = f(n)$. Then $(m, m) = (n, n)$, so $m = n$.

(k) f is one-to-one. ***Proof.*** Suppose $x, z \in [2, 3)$ and $f(x) = f(z)$. Then $\dfrac{x-2}{3-x} = \dfrac{z-2}{3-z}$, so $3x - xz - 6 + 2z = 3z - xz - 6 + 2x$. Thus $x = z$.

4. (a) $B = \{0, 3\}$, $f = \{(1, 0), (2, 3), (3, 0), (4, 0)\}$

8. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$ and $g: \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$. Then f maps onto $\mathbb{R}$ but $g \circ f$ is not onto $\mathbb{R}$.

(e) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, and $C = \{x, y, z\}$. The function f must not be one-to-one. Let $f = \{(a, 2), (b, 2), (c, 3)\}$ and $g = \{(1, x), (2, y), (3, z)\}$. Then g is one-to-one, but the composite $g \circ f = \{(a, y), (b, y), (c, z)\}$ is not.

9. (c) ***Proof.*** We verify that f maps onto $\mathbb{R}$ as follows: $1 \in \text{Rng}(f)$, because $f(4) = 1$. For $w \neq 1$, choose $x = \dfrac{4w+2}{1-w}$. Then $x \neq -4$ and

$$f(x) = \left[\frac{4w+2}{1-w} - 2\right] \div \left[\frac{4w+2}{1-w} + 4\right] = \frac{6w}{6} = w.$$

Therefore, f maps onto $\mathbb{R}$.

To show that f is one-to-one, suppose $f(x) = f(z)$. Then for $x \neq -4$ and $z \neq -4$, $\dfrac{x-2}{x+4} = \dfrac{z-2}{z+4}$. Therefore, $xz - 2z + 4x - 8 = xz - 2x + 4z - 8$, so $6x = 6z$ and $x = z$. We must also consider whether $f(x)$ might be 1, for $x \neq -4$. But if $x \neq -4$ and $f(x) = \dfrac{x-2}{x+4} = 1$, then $x - 2 = x + 4$, so $-2 = 4$. This is impossible. Therefore f is one-to-one.

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11. **(a)** ***Proof.*** f is not a surjection because $[1]$ has no pre-image in $\mathbb{Z}_4$. This is because if $f(\bar{x}) = [1]$, then $[2x] = [1]$. But then 8 divides the odd number $2x - 1$, which is impossible. To show f is an injection, suppose $f(\bar{x}) = f(\bar{z})$. Then $[2x] = [2z]$, so $2x = 2z \pmod 8$. Therefore, 8 divides $2x - 2z$, so 4 divides $x - z$, and thus $\bar{x} = \bar{z}$.

12. **(b)** Example 1: $x_n = n$ for all $n \in \mathbb{N}$ (the identity function).

Example 2: $x_n = \begin{cases} 50 - n & \text{if } n < 50 \\ n & \text{if } n \geq 50 \end{cases}$.

Example 3: $x_n = \begin{cases} n + 1 & \text{if } n \text{ is odd} \\ n - 1 & \text{if } n \text{ is even} \end{cases}$.

The sequence of example 3 is 2, 1, 4, 3, 6, 5, 8, 7, . . .

13. **(c)** None.

(f) Since $m = n + 1$, one element of B has two pre-images. This element of B can be selected in n ways, and the two pre-images in $\binom{n+1}{2}$ ways. We can assign each of the remaining $n - 1$ elements of B as the image of exactly one of the $n - 1$ remaining elements of A in $(n - 1)!$ ways. Thus there are $n\binom{n+1}{2}(n-1)! = n!\binom{n+1}{2}$ functions from A onto B.

14. **(a)** F. To show that f is onto $\mathbb{R}$ we must prove that $\mathbb{R} \subseteq \operatorname{Rng}(f)$. The proof shows only that $\operatorname{Rng}(f) \subseteq \mathbb{R}$.

(b) *Hint:* What additional information should be included in this proof?

(d) A. Notice that a direct proof would have been a little easier to follow.

Exercises 4.4

3. **(b)** Let h be the inverse of g. Then

$$\begin{aligned}(x, y) \in h \text{ iff } (y, x) \in g \\ &\text{iff } x = \frac{4y}{y+2} \\ &\text{iff } xy + 2x = 4y \\ &\text{iff } y(x - 4) = -2x \\ &\text{iff } y = \frac{2x}{4 - x}, \text{ for } x < 4.\end{aligned}$$

Therefore $h(x) = \dfrac{2x}{4 - x}$. To verfiy that this formula is correct, suppose $x > -2$. Then

$$\begin{aligned}(h \circ g)(x) &= h(g(x)) \\ &= h\left(\frac{4x}{x+2}\right) \\ &= \frac{2\left(\frac{4x}{x+2}\right)}{4 - \frac{4x}{x+2}}\end{aligned}$$

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$$= \frac{\frac{8x}{x+2}}{\frac{4(x+2)-4x}{x+2}}$$
$$= x.$$

Since the composite is the identity function on the domain of g, we conclude that $h = g^{-1}$.

9. **(c)** [2 5 4 6 1 7 3]
(e) [6 1 4 7 5 3 2]
(i) [5 3 7 4 1 2 6]

Exercises 4.5

1. **(a)** $h(\varnothing) = \varnothing$, $h(\{1\}) = \{4\}$, $h(\{2\}) = \{4\}$, $h(\{3\}) = \{5\}$, $h(\{1, 2\}) = \{4\}$, $h(\{1, 3\}) = \{4, 5\}$, $h(\{2, 3\}) = \{4, 5\}$, $h(A) = \{4, 5\}$.

2. **(a)** $[2, 10]$
(c) $\{0\}$

4. **(b)** $[2, 5.2]$
(c) $\left[2 - \sqrt{3}, \dfrac{3 - \sqrt{5}}{2}\right) \cup \left(\dfrac{3 + \sqrt{5}}{2}, 2 + \sqrt{3}\right]$

8. **(a)** Suppose $b \in f\left(\bigcap_{\alpha\in\Delta} D_\alpha\right)$. Then $b = f(a)$ for some $a \in \bigcap_{\alpha\in\Delta} D_\alpha$. Thus $a \in D_\alpha$ for every $\alpha \in \Delta$. Since $b = f(a)$ we conclude that $b \in f(D_\alpha)$ for every $\alpha \in \Delta$. Therefore, $b \in \bigcap_{\alpha\in\Delta} f(D_\alpha)$. This proves that $f\left(\bigcap_{\alpha\in\Delta} D_\alpha\right) \subseteq \bigcap_{\alpha\in\Delta} f(D_\alpha)$.

9. *Hint:* There must be at least two sets D_1 and D_2 in the family, and $b \in f(D_\alpha)$ for all $\alpha \in \Delta$, but no element a in $\bigcap_{\alpha\in\Delta} D_\alpha$ such that $f(a) = b$. The function f cannot be one-to-one.

10. **(a)** Suppose $b \in f(f^{-1}(E))$. Then there is $a \in f^{-1}(E)$ such that $f(a) = b$. Since $a \in f^{-1}(E)$, $f(a) \in E$. But $f(a) = b$, so $b \in E$. Therefore, $f(f^{-1}(E)) \subseteq E$.

(d) First, suppose $E = f(f^{-1}(E))$. Suppose $b \in E$. Then $b \in f(f^{-1}(E))$. Thus there is $a \in f^{-1}(E)$ such that $b = f(a)$, so $b \in \text{Rng}(f)$. Therefore, $E \subseteq \text{Rng}(f)$.

Now assume $E \subseteq \text{Rng}(f)$. We know by part (a) that $f(f^{-1}(E)) \subseteq E$, so to prove equality, we must show $E \subseteq f(f^{-1}(E))$. Suppose $b \in E$. Then $b \in \text{Rng}(f)$, so $b = f(a)$ for some $a \in A$. Since $b = f(a) \in E$, $a \in f^{-1}(E)$. Thus $b = f(a)$ and $a \in f^{-1}(E)$, so $b \in f(f^{-1}(E))$. Therefore, $E \subseteq f(f^{-1}(E))$.

11. **(b)** **Proof.** Suppose $t \in f(X) - f(Y)$. Then $t \in f(X)$, so there exists $x \in X$ such that $f(x) = t$. We note $x \notin Y$ since $t = f(x) \notin f(Y)$. Thus $x \in X - Y$ and therefore $t = f(x) \in f(X - Y)$.

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12. The converse is true. To prove f is one-to-one, suppose $x, y \in A$ and $x \neq y$. Then $\{x\} \cap \{y\} = \varnothing$ and thus $f(\{x\} \cap \{y\}) = \varnothing$. By hypothesis, $f(\{x\} \cap \{y\}) = f(\{x\}) \cap f(\{y\}) = \{f(x)\} \cap \{f(y)\}$. Thus $f(x) \neq f(y)$.

15. **(a)** If f is one-to-one, then the induced function is one-to-one.

18. **(a)** F. The claim is not true. We cannot conclude $x \in X$ from $f(x) \in f(X)$.

Exercises 4.6

3. **(a)** does not exist

(c) $\frac{4}{5}$

(e) 0

(g) e^2 (Recall that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.)

(i) 0

5. **(a)** *Hint:* To show x diverges, suppose the limit is L and let $\varepsilon = 1$.

(c) *Hint:* x diverges; use $\varepsilon = 1$.

(f) *Hint:* $x_n \to 0$; for $\varepsilon > 0$, use $N > (2\varepsilon)^{-2}$ and

$$\sqrt{n+1} - \sqrt{n} = \left(\sqrt{n+1} - \sqrt{n}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right).$$

6. **(a)** **Proof.** Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. Since $x_n \to L$, there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$, then $|x_n - L| < \frac{\varepsilon}{2}$. Likewise, there exists $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|y_n - M| < \frac{\varepsilon}{2}$. Let $N_3 = \max\{N_1, N_2\}$, and assume $n > N_3$. Therefore we have $|(x_n + y_n) - (L + M)| = |(x_n - L) + (y_n - M)| \leq |x_n - L| + |y_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore, $x_n + y_n \to L + M$.

(f) *Hint:* $|x_n| - |L| \leq |x_n - L|$.

7. **(a)** *Hint:* Since $x_n \to L$, $|x_n| \to |L|$, by Exercise 6(f). Now apply the definition of $|x_n| \to |L|$ with $\varepsilon = \frac{|L|}{2}$.

10. **(b)** A. The proof uses Exercise 6(b).

Exercises 5.1

2. **(a)** finite

(c) finite

(e) infinite

(f) finite

6. **(a)** Suppose A is finite. Since $A \cap B$ is a subset of A, $A \cap B$ is finite.

7. *Hint:* Write $A \cup B = (A - B) \cup B$ and $A = (A - B) \cup (A \cap B)$ and apply Theorem 5.1.7(a).

9. **(a)** *Hint:* Define $f\colon A \to A \cup \{x\}$ by $f(a) = (a, x)$, for each $a \in A$. Now show that f is one-to-one and onto $A \cup \{x\}$.

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10. *Hint:* First show $\mathcal{P}(A \times B)$ is finite.

11. **(c)** not possible

13. Obviously $\mathbb{N}_r - \{x\} \approx \mathbb{N}_{r-1}$ when $x = r$. If $x \neq r$, define a function on $\mathbb{N}_r - \{x\}$ by considering first the images of elements that are less than x and then images of elements greater than x.

15. *Hint:* Use the argument that if $n < m$, then the finite set $\mathbb{N}_m$ is equivalent to one of its proper subsets.

18. **(a)** *Hint:* Suppose f is not onto B and consider the range of f.

(b) *Hint:* Suppose f is not one-to-one. Then A is not empty and since A and B are finite and $A \approx B$, there is some $n \in \mathbb{N}$ such that $\mathbb{N}_n \approx A$ and $B \approx \mathbb{N}_n$. Use these facts to construct a function F from $\mathbb{N}_n$ onto $\mathbb{N}_n$ that is not one-to-one. Then for some $x, y, z \in \mathbb{N}_n$, $f(x) = f(y) = z$. Removing (y, z) from F produces a function from a proper subset of $\mathbb{N}_n$ onto $\mathbb{N}_n$. Now apply Exercise 17.

19. *Hint:* Use induction on the number of elements in the domain.

20. ***Proof.*** Suppose A and B are finite, $\overline{\overline{A}} = m$, $\overline{\overline{B}} = n$, $m > n$, and $f\colon A \to B$ is one-to-one. Then there exists a one-to-one (and onto) function $g\colon \mathbb{N}_m \to A$ and a one-to-one (and onto) function $h\colon B \to \mathbb{N}_n$. See the diagram below. Then $h \circ f \circ g$ is a one-to-one function from $\mathbb{N}_m$ to $\mathbb{N}_n$ by Theorem 4.3.4. But $m > n$ so the conclusion that $h \circ f \circ g$ is a one-to-one contradicts Theorem 5.1.9.

$$\begin{tikzcd}
 & A \arrow[r, "f"] & B \arrow[dr, "h"] & \\
\mathbb{N}_m \arrow[ur, "g"] \arrow[rrr, "h \circ f \circ g"'] & & & \mathbb{N}_n
\end{tikzcd}$$

21. **(b)** *Hint:* The largest possible sum of 10 elements of $\mathbb{N}_{99}$ is $90 + 91 + \cdots + 99 = 945$. Thus there are no more than 945 possible sums. However, there are $2^{10} - 1 = 1{,}023$ nonempty subsets of S. Apply the Pigeonhole Principle, and delete any common elements from two subsets that have the same sum to form disjoint subsets.

22. **(b)** C. In Case 2, it is not correct that $\mathbb{N}_k \cup \mathbb{N}_1 \approx \mathbb{N}_{k+1}$. In fact, $\mathbb{N}_k \cup \mathbb{N}_1 = \mathbb{N}_k$.

Exercises 5.2

3. **(a)** ***Proof.*** Let $f\colon \mathbb{N} \to D^+$ be given by $f(n) = 2n - 1$ for each $n \in \mathbb{N}$. We show that f is one-to-one and maps onto D^+. First, to show f is one-to-one, suppose $f(x) = f(y)$. Thus $2x - 1 = 2y - 1$, which implies $x = y$. Also, f maps onto D^+ since if d is an odd positive integer, then d has the form $d = 2r - 1$ for some $r \in \mathbb{N}$. But then $f(r) = d$.

(e) *Hint:* Consider $f(x) = -(x + 12)$ with domain $\mathbb{N}$.

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4. **(a)** *Hint:* Let $f(0, 1) \to (1, \infty)$ be given by $f(x) = \frac{1}{x}$.

(b) *Hint:* Let $f(0, 1) \to (a, \infty)$ be given by $f(x) = (a-1) + \frac{1}{x}$.

(d) *Hint:* Let $f(0, 1) \to [1, 2) \cup (5, 6)$ be given by

$$f(x) = \begin{cases} 2 - 2x & \text{if } 0 < x \le \frac{1}{2} \\ 2x + 4 & \text{if } \frac{1}{2} < x < 1 \end{cases}.$$

6. **(a)** Define $g\colon \mathbb{N} \to E^+$ by

$$g(x) = \begin{cases} 20 & \text{if } x = 1 \\ 2 & \text{if } x = 10 \\ 2x & \text{if } x \ne 1, x \ne 10 \end{cases}.$$

7. **(a)** **c**

(c) $\aleph_0$

(e) **c**

12. **(a)** F. W is certainly an infinite subset of $\mathbb{N}$, and D^+ is denumerable, but this "proof" claims without justification that every infinite subset of $\mathbb{N}$ is denumerable. To show W is denumerable, we need to use another theorem or a bijection between W and a denumerable set.

(c) F. The claim is false. Also "A and B are finite" is not a denial of "A and B are infinite."

(d) F. Writing an infinite set A as $\{x_1, x_2, \ldots\}$ is the same as assuming A is denumerable.

Exercises 5.3

6. *Hint:* Use a proof by induction on n, the number of sets in the family.

7. *Hint:* This theorem has been proved in the cases where A and B are finite (Theorem 5.1.7(a)), where one set is denumerable and the other is finite (Theorem 5.3.4), and where A and B are denumerable and disjoint (Theorem 5.3.5). The only remaining case is where A and B are denumerable and not disjoint. Write $A \cup B$ as $A \cup (B - A)$, a union of disjoint sets. Now explain why $B - A$ is countable, and then apply Theorems 5.3.4 and 5.3.5.

8. *Hint*: Give a proof by induction.

11. *Hint:* For each $m \in \mathbb{N}$, let $\overline{\overline{B}}_m = k_m$. Then there is a bijection $f_m\colon B_m \to \mathbb{N}_{k_m}$. Define $h\colon \bigcup_{i\in\mathbb{N}} B_i \to \mathbb{N}$ by $h(x) = \left(\sum_{i=1}^{m-1} k_i\right) + f_m(x)$, for $x \in B_m$.

13. **(b)** *Hint:* First prove that each set T_n is infinite. Then for $a \in T_n$ define $f(a) = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot \cdots p_k^{a_k}$, where p_k is the kth prime and $a_i = 0$ for all $i > k$. Explain why T_n is equivalent to $\text{Rng}(f)$ and why $\text{Rng}(f)$ is countable.

15. **(a)** C. The proof is valid only when $f(1) = x$. In the case when $f(1) \ne x$, we need a new function g that is almost the same as f except that the

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image of 1 will be x. This involves removing the two ordered pairs with second coordinates x and $f(1)$ and replacing them with two other ordered pairs. Let t be the unique element of $\mathbb{N}$ such that $f(t) = x$ and define $f^* = (f - \{(1, f(1)), (t, x)\}) \cup \{(1, x), (t, f(1))\}$. Now let $g(n) = f^*(n + 1)$ for all $n \in \mathbb{N}$.

Exercises 5.4

2. *Hint:* Recall that $\overline{\overline{\mathcal{P}(\mathbb{N})}} = \overline{\overline{\mathbb{R}}}$.

5. (b) true

8. (a) $\overline{\overline{\varnothing}} < \overline{\overline{\{0\}}} < \overline{\overline{\{0, 1\}}} < \overline{\overline{\mathbb{Q}}} < \overline{\overline{(0, 1)}} = \overline{\overline{[0, 1]}} = \overline{\overline{\mathbb{R} - \mathbb{N}}}$

$$= \overline{\overline{\mathbb{R}}} < \overline{\overline{\mathcal{P}(\mathbb{R})}} < \overline{\overline{\mathcal{P}(\mathcal{P}(\mathbb{R}))}}.$$

11. (b) not possible

16. (a) *Hint:* Assume that there is a bijection. Associate each function f in $\mathcal{F}$ with the corresponding real number a such that $0 \le a \le 1$, and write f as f_a. Define $g: [0, 1] \to [0, 1]$ by

$$g(x) = \begin{cases} 0 & \text{if } f_x(x) \neq 0 \\ 1 & \text{if } f_x(x) = 0 \end{cases}.$$

Then $g = f_b$ for some $b \in [0, 1]$. Compute $g(b)$ to obtain a contradiction. [Ref.: J. Robertson, "A Student Exercise on Cardinality," *Mathematics and Computer Education* 32 (1998), 17–18.]

(b) *Hint:* Consider the set of constant functions in $\mathcal{F}$.

17. (b) F. The claim is false. We have not defined or discussed properties of operations such as addition for infinite cardinal numbers. Thus the equation $\overline{\overline{C}} = \overline{\overline{B}} + (\overline{\overline{C - B}})$ applies only to finite sets.

(d) F. The "proof" assumes that every element of B is in the range of f.

Exercises 5.5

1. (b) The Axiom of Choice is not necessary since there are only a finite number of sets in the collection.

(h) The Axiom of Choice is necessary.

5. ***Proof.*** Let $B \subseteq A$ with B infinite and A denumerable. Since $B \subseteq A$, $\overline{\overline{B}} \le \overline{\overline{A}}$. Since A is denumerable, $\overline{\overline{A}} = \overline{\overline{\mathbb{N}}}$. Since B is infinite, B has a denumerable subset D by Theorem 5.5.4. Thus $\overline{\overline{A}} = \overline{\overline{\mathbb{N}}} = \overline{\overline{D}} \le \overline{\overline{B}}$. By the Cantor–Schröder–Bernstein Theorem, $\overline{\overline{B}} = \overline{\overline{A}}$. Thus $B \approx A$.

Alternate Proof. The set B is an infinite subset of the countable set A, so B is countable and denumerable. Since both A and B are equivalent to $\mathbb{N}$, A is equivalent to B.

8. *Hint:* Let $x \in A$. By Theorem 5.5.4, $A - \{x\}$ has a denumerable subset $\{a_n : n \in \mathbb{N}\}$. Construct a one-to-one correspondence between A and $A - \{x\}$.

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10. **(a)** A. Note that the range of every sequence is countable.

(c) F. The idea of this "proof" is to take out countably many elements, one at a time, until denumerably many elements are left. But if A is uncountable and C is countable, then the set $B = A - C$ of leftover elements will always be uncountable. (See Exercise 9(b) of Section 5.3.)

Exercises 6.1

1. **(a)** yes

(e) no

2. **(a)** not commutative

(e) not an operation

3. **(a)** not associative

(e) not an operation

4. **(a)** a is the identity element

(b) Yes. This is tedious to verify because one must verify that 64 equations of the form $(x \circ y) \circ z = x \circ (y \circ z)$ are all true. It helps to observe that if $x = a$ (the identity) then $(x \circ y) \circ z = y \circ z = x \circ (y \circ z)$. Similarly, if $y = a$ or $z = a$, the equation is easily seen to be true. This leaves only 27 cases to verify when none of x, y, or z is a. For example, $(b \circ c) \circ b = d \circ b = c$, while $b \circ (c \circ b) = (b \circ d) = c$, so the equation is true when $x = b$, $y = c$, and $z = b$.

(c) Yes, because the table is symmetric about its main diagonal. To verify by cases that the equation $x \circ y = y \circ x$ is true for every choice of x and y, consider first that case that one of x or y is a, then the case that $x = y$, and finally the other 3 cases.

(d) The inverses of a, b, c, d, are a, b, c, d, respectively.

(e) No. The product $b \circ c = d$ is not in B_1, so B_1 is not closed under $\circ$.

(f) Yes. $a \circ a = a$, $a \circ c = c$, $c \circ a = c$, $c \circ c = a$.

(g) $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, and $\{a, b, c, d\}$.

(h) True. In fact, for all $x \in A$, $x \circ x = a$.

8. *Hint:* Compute $(ac)(db)$ and $(db)(ac)$.

9. **(a)** *Hint:* Compute $x \circ (a \circ y)$ and $(x \circ a) \circ y$.

10. **(b)** *Hint:* Assume that for some natural number n, every product of t elements $a_1, a_2, \ldots, a_t$ of A is equal to $(\ldots((a_1 * a_2) * a_3)\ldots) * a_t$ for every $t \le n$. Now consider a product of $n + 1$ factors $a_1, a_2, \ldots, a_{n+1}$ in that order. This product has the form $b_1 * b_2$, where b_1 is a product of some k factors $(k \le n)$ $a_1, a_2, \ldots, a_k$ in that order and b_2 is a product of the remaining factors $a_{k+1}, \ldots, a_{n+1}$ in that order. First use the induction hypothesis to write b_1 and b_2 in left-associated form.

Now consider two cases. If $k > 1$, then there are at least two factors a_1 and a_2 in the product b_1. Denote the product $a_1 * a_2$ by c, which is an element of A. Replace $a_1 * a_2$ by c and use the hypothesis of induction to write $b_1 * b_2$ as a left-associated product.

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If $k = 1$ and b_2 has only one factor, then the product $b_1 * b_2$ is $a_1 * a_2$ which is already in left-associated form. Otherwise, b_2 has at least 2 factors, and we may denote by d the product of the first two factors a_2 and a_3 of b_2. Replace $a_2 * a_3$ by d in the product $b_1 * b_2$ and apply the hypothesis of induction. Finally, apply the associative property to $a_1 * (a_2 * a_3)$ to write the entire product in left-associated form.

12. *Hint*: Assume that $a = c \pmod{m}$ and $b = d \pmod{m}$. Then there exist integers x and y such that $mx = a - c$ and $my = b - d$. To show that $a + b = c + d \pmod{m}$, compute the sum of mx and my. To show that $a \cdot b = c \cdot d \pmod{m}$, simplify $bmx + cmy$.

15. **(a)** 2, 4, and 6.

(c) There are no divisors of zero.

16. **(b)** F. The claim is false. One may premultiply (multiply on the left) or postmultiply both sides of an equation by equal quantities. Multiplying one side on the left and the other on the right does not always preserve equality.

(d) F. The proof makes the assumption that $xy \neq 0$, which may be false.

Exercises 6.2

1. **(c)**

$\cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

We see from the table that the set is closed under $\cdot$, 1 is the identity, and each element has an inverse. Also, $\cdot$ is associative.

(d) *Hint:* $\emptyset$ is the identity.

2.

	e	u	v	w
e	e	u	v	w
u	u	v	w	e
v	v	w	e	u
w	w	e	u	v

4. **(c)** The group is abelian.

8. **(b)** [3 2 1 4], [1 2 3 4], [2 4 1 3].

12. *Hint:* For a, $b \in G$, compute a^2b^2 and $(ab)^2$ two ways.

13. *Hint:* To have both cancellation properties, every element must occur in every row and in every column of the table.

16. **(a)** v, w, e, and u

(b) *Hint:* Let $a, b \in G$. For an element x such that $a * x = b$, try $a^{-1} * b$.

19. Since $(p - 1)(p - 1) = p^2 - 2p + 1 = p(p - 2) + 1$, $(p - 1)^2 = 1 \pmod{p}$. Therefore $(p - 1)(p - 1) = 1$ in $\mathbb{Z}_p$, and hence $(p - 1)^{-1} = p - 1$.

20. **(a)** $x = 0, 4, 8, 12, 16$

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21. **(c)** $x = 6$
(e) No solution
22. **(a)** $(x-1)(x+1) = x^2 - 1 = 0$
23. **(b)** F. A minor criticism is that no special case is needed for e. The fatal flaw is the use of the undefined division notation.
(c) A. The proof is correct, but provides minimal explanation.

Exercises 6.3

1. **(a)** $\{0\}, \mathbb{Z}_8, \{0, 4\}, \{0, 2, 4, 6\}$
(d) *Hint:* There are six subgroups.
7. **(a)** Yes. Assume G is abelian and H is a subgroup of G. Suppose $x, y \in H$. Then $x, y \in G$. Therefore, $xy = yx$.
9. **(c)** The order of 0 is 1. The elements 1, 3, 5, and 7 have order 8. The elements 2 and 6 have order 4. The order of 4 is 2.
11. **Proof.** The set C_a is not empty because $ea = a = ae$, so $e \in C_a$. Let $x, y \in C_a$. Then $xa = ax$ and $ya = ay$. Multiplying both sides of the last equation by y^{-1}, we have $y^{-1}(ya)y^{-1} = y^{-1}(ay)y^{-1}$. Thus $(y^{-1}y)(ay^{-1}) = (y^{-1}a)(yy^{-1})$, or $ay^{-1} = y^{-1}a$. Therefore, $(xy^{-1})a = x(y^{-1}a) = x(ay^{-1}) = (xa)y^{-1} = (ax)y^{-1} = a(xy^{-1})$. This shows $xy^{-1} \in C_a$. Therefore, C_a is a subgroup of G, by Theorem 6.3.3.
14. **Proof.** The identity $e \in H$ because H is a group, and thus $a^{-1}ea \in K$. Thus K is not empty. Suppose $b, c \in K$. Then $b = a^{-1}h_1a$ and $c = a^{-1}h_2a$ for some $h_1, h_2 \in H$. Thus $bc^{-1} = (a^{-1}h_1a)(a^{-1}h_2a)^{-1} = (a^{-1}h_1a)(a^{-1}h_2^{-1}a) = a^{-1}h_1(aa^{-1})h_2^{-1}a = a^{-1}h_1h_2^{-1}a$. But H is a group, so $h_1h_2^{-1} \in H$. Thus $bc^{-1} \in K$. Therefore, K is a subgroup of G.
17. *Hint*: Let H be a subgroup of (a). If H is the trivial subgroup it is clearly cyclic. Otherwise show that there is a positive integer t such that a^t is in H. Now use the Well Ordering Principle to find the smallest such t and use the Division Algorithm to show that this power of a is a generator for H.
18. **(c)** a^{10}, a^{20}
19. **(a)** C. The proof omits the step of verifying that $H \cap G$ is nonempty (because it contains the identity).

Exercises 6.4

4. Let $f, g \in F$. Then $I(f+g) = \int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = I(f) + I(g)$.
8. **(a)** Let $C, D \in \mathcal{P}(A)$. Then $f(C \cup D) = f(C) \cup f(D)$ by Theorem 4.5.1(b). Therefore, f is an OP mapping.
10. **(a)** *Hint*: Suppose $G = (\{e, a\}, \circ)$ and $H = (\{i, b\}, *)$ are two groups with identity elements e and i. To define an isomorphism from G to H, first determine the image of e.
(c) *Hint*: Use Theorem 6.4.3(c) to show that the algebraic system in Exercise 2 of Section 6.2 is not isomorphic to $(\mathbb{Z}_4, +)$.

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14. **(a)** Suppose $\bar{y} = \bar{x}$ in $\mathbb{Z}_{18}$. Then 18 divides $x - y$. Therefore, 6 divides $x - y$, and so 24 divides $4(x - y) = 4x - 4y$. Thus $f(\bar{x}) = [4x] = [4y] = f(\bar{y})$ in $\mathbb{Z}_{24}$, so f is well defined. Now let $\bar{x}, \bar{y} \in \mathbb{Z}_{18}$. Then $f(\bar{x} +_{18} \bar{y}) = [4(x + y)] = [4x + 4y] = [4x] +_{24} [4y] = f(\bar{x}) +_{24} f(\bar{y})$.

(b) $\text{Rng}(f) = \{[0], [4], [8], [12], [16], [20]\}$. The table is:

	[0]	[4]	[8]	[12]	[16]	[20]
[0]	[0]	[4]	[8]	[12]	[16]	[20]
[4]	[4]	[8]	[12]	[16]	[20]	[0]
[8]	[8]	[12]	[16]	[20]	[0]	[4]
[12]	[12]	[16]	[20]	[0]	[4]	[8]
[16]	[16]	[20]	[0]	[4]	[8]	[12]
[20]	[20]	[0]	[4]	[8]	[12]	[16]

Exercises 6.5

1. **(b)** The interval $[-1, 1]$ is not a ring because it is not closed under addition. (For example, $\frac{3}{4} + \frac{3}{4}$ in not an element of $[-1, 1]$).

4. *Hint:* First show that $\mathbb{Z} \times \mathbb{Z}$ is closed under $\oplus$ and $\otimes$ and that the additive identity is $(0, 0)$. The inverse of (a, b) is $(-a, -b)$ because

$$(a, b) \oplus (-a, -b) = (a + (-a), b + (-b)) = (0, 0), \text{ and}$$
$$(-a,-b) \oplus (a, b) = (-a + a, - b + b) = (0, 0).$$

Explain why the remaining axioms hold for $\mathbb{Z} \times \mathbb{Z}$.

13. **(a)** *Hint*: Let $m \in \mathbb{N}$. First, show that if m is composite, then m has divisors of 0. Now suppose m is prime and there are nonzero elements a and b in $\mathbb{Z}_m$ such that $ab = 0$. Apply Euclid's Lemma to reach a contradiction.

Exercises 7.1

1. **(b)** $\frac{1}{3}$, 1, 2, 3

2. **(b)** 0 and all negative real numbers are lower bounds.

3. **(a)** supremum: 1; infimum: 0

(c) supremum: does not exist; infimum: 0

(e) supremum: 1; infimum: $\frac{1}{3}$

(g) supremum: 5; infimum: -1

4. **(a)** *Hint*: Show that an upper bound for A is an upper bound for B.

6. *Hint:* Suppose b is an upper bound for A. Therefore, for all x, if $x \in A$ then $x \leq b$. This means that for all x, if $x > b$, then $x \notin A$. Explain why A^c is not bounded above.

8. **(a)** **Proof.** Let x and y be least upper bounds for A. Then x and y are upper bounds for A. Since y is an upper bound and x is a least upper bound,

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$x \le y$. Since x is an upper bound and y is a least upper bound, $y \le x$. Thus $x = y$.

13. **(a)** **Proof.** Let $s = \sup(A)$, $B = \{u: u \text{ is an upper bound for } A\}$. Then B is bounded below (by elements of A) so $\inf(B)$ exists. Let $t = \inf(B)$. We must show $s = t$.

(i) To show $t \le s$ we note that since $s = \sup(A)$, s is an upper bound for A. Thus $s \in B$. Therefore, $t \le s$.

(ii) To show $s \le t$ we will show t is an upper bound for A. If t is not an upper bound for A, then there exists $a \in A$ with $a > t$. Let $\varepsilon = \frac{a-t}{2}$. Since $t = \inf(B)$ and $t < t + \varepsilon$, there exists $u \in B$ such that $u < t + \varepsilon$. But $t + \varepsilon < a$. Therefore, $u < a$, a contradiction, since $u \in B$ and $a \in A$.

14. **(a)** **Proof.** Suppose $\sup(A)$ and $\sup(B)$ exist. Then $A \cup B$ is bounded above by $m = \max\{\sup(A), \sup(B)\}$ (see Exercise 4(c)). By the completeness property, $\sup(A \cup B)$ exists. We show that $\sup(A \cup B) = m$.

(i) Since $A \subseteq A \cup B$, we have $\sup(A) \le \sup(A \cup B)$. Also, $B \subseteq A \cup B$ implies that $\sup(B) \le \sup(A \cup B)$. It follows that $m = \max\{\sup(A), \sup(B)\} \le \sup(A \cup B)$.

(ii) It suffices to show m is an upper bound for $A \cup B$. Let $x \in A \cup B$. If $x \in A$, then $x \le \sup(A) \le m$. If $x \in B$, then $x \le \sup(B) \le m$. Thus m is an upper bound for $A \cup B$. Hence $\sup(A \cup B) \le m$.

19. *Hint:* Let F be an ordered field. Assume that F is complete and let A be a nonempty subset of F that has a lower bound in F. To show that A has an infimum in F, begin by defining the set $A^- = \{-x: x \in A\}$. Prove that A^- has a supremum and then find an infimum for A.

21. **(a)** F. The claim is true but $y = i + \frac{\varepsilon}{2}$ might not be in A.

Exercises 7.2

1. **(b)** $x = 3.825$, $\delta = 0.025$

2. **(b)** *Hint:* In the case where $|x_2 - x_1| < 2\delta_1$,

$$\mathcal{N}(x_1, \delta_1) \cap \mathcal{N}(x_2, \delta_1) = \mathcal{N}\left(\frac{x_1 + x_2}{2}, \delta_1 - \frac{|x_2 - x_1|}{2}\right).$$

3. $\lim_{x \to a} f(x) = L$ iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathcal{N}(a, \delta)$, then $f(x) \in \mathcal{N}(L, \varepsilon)$.

4. **(e)** $\varnothing$

(i) $\bigcup_{n \in \mathbb{N}} (n + 0.1, n + 0.2)$

5. **(b)** open

(e) open

(i) closed

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13. **(a)** not compact (not bounded)
(d) not compact (neither closed nor bounded)
(g) not compact (not closed)
15. **(a)** *Hint:* First show that a cover for $A \cup B$ is a cover for A and a cover for B.
19. **(c)** C. With the addition of O^* to the cover $\{O_\alpha\colon \alpha \in \Delta\}$ we are assured that there is a finite subcover of $\{O^*\} \cup \{O_\alpha\colon \alpha \in \Delta\}$, but not necessarily a subcover of $\{O_\alpha\colon \alpha \in \Delta\}$. Since $O^* = A - B$ is useless in a cover for B, it can be deleted from the subcover.

Exercises 7.3

1. (c) *Hint*: Use the fact the $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.
3. **(a)** $\left\{\frac{1}{2}\right\}$
(c) $\varnothing$
(g) $\{0, 2\}$
(m) $\mathbb{N}$
4. $\{0, 1\}$
5. *Hint:* Let $a \in A$. Then $z > a$. Show z is an accumulation point of A by using Theorem 7.1.1.
7. **(b)** *Hint:* Use Exercise 6(a).
10. **(a)** The set has no accumulation points.
(c) The set has at least one accumulation point.
13. **(a)** F. *Hint:* Identify the misuse of quantifiers.
(c) F. The claim is false. $(B^c)'$ need not be a subset of $(B')^c$.

Exercises 7.4

1. **(a)** bounded below by 10, not bounded above
(c) bounded; bounded above by $\frac{1}{10}$, bounded below by 0
(e) bounded; bounded above by 10, bounded below by 0
(g) not bounded above, not bounded below
(i) bounded; bounded above by 0.81, bounded below by -0.9
3. *Hint:* Let y be bounded, and B a number such that $|y_n| \le B$ for all $n \in \mathbb{N}$. Use the definition of $x_n \to 0$ with $\frac{\varepsilon}{B}$.
5. **(f)** *Hint:* It suffices to show that for $n \in \mathbb{N}$, $\dfrac{(n+1)!}{(n+1)^{n+1}} < \dfrac{n!}{n^n}$.

Exercises 7.5

3. **(a)** *Hint:* Consider rational numbers in $[7, 8]$ whose square is less than 50.
6. **(a)** *Hint:* Consider a set A that includes $[0, 2]$ and the sequence $x_n = n/(n+1)$.

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(b) F. The claim is correct but there is little that is correct in this proof. For instance, the upper bound a_0 for A may be negative, in which case B would have to be defined differently. The most serious error is that there is no connection between being an accumulation point and being an upper bound for a set.

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