Math 135B, Winter 2023.

Homework 1

Due: Jan. 20, 2023

1. (a) Assume that X_n are random variables such that that $X_n \to a$ in probability, where a is a (nonrandom) constant. Suppose also $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Show that $f(X_n) \to f(a)$ in probability.

(b) Assume that X_n and Y_n are random variables such that X_n and Y_n are defined on the same probability space (so that they can be added). Assume also that $X_n \to a$ and $Y_n \to b$ in probability, where a and b are (nonrandom) constants. Show that $X_n + Y_n \to a + b$ in probability.

2. Assume that $X_1, X_2, \dots$ are independent random variables, all uniform on $[0, 1]$. Compute the limit, in probability, of the following random variables:

(a) $\frac{1}{n} \sum_{i=1}^{n} X_n$;

(b) $\frac{1}{n} \sum_{i=1}^{n} X_n^2$;

(c) $\frac{1}{n} \sum_{i=1}^{n} X_i X_{i+1}$; and

(d) $(X_1 \cdots X_n)^{1/n}$.

3. Assume you have $2n$ cards with n colors, with 2 cards of each color. Select n cards without replacement, and let N_n be the number of colors that *are not* represented in your selection.

(a) Compute EN_n and $\mathrm{Var}(N_n)$.

(b) Determine a constant c so that $\frac{1}{n} N_n \to c$, in probability.

(c) Let M_n be the the number of colors that *are* represented in your selection. Determine a constant d so that $\frac{1}{n} M_n \to d$, in probability.

4. (From a Final Exam at Queen's University, Ontario.) An urn contains m red and n blue balls. Balls are drawn one at a time withour replacement until all m red balls are drawn. Let T be the number of draws required. Compute ET. (*Hint*. The best way is to relate T to the number N of blue balls that *remain* in the urn after all red balls are drawn.)