Math 135B, Winter 2023.

Homework 2

Due: Jan. 27, 2023

1. Assume that $X_1, \ldots, X_5$ are independent random variables with the same density

$$f(x) = \begin{cases} \frac{1}{e-1} e^x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the moment generating function of X_1.

(b) Compute the moment generating function of $S = X_1 + \ldots + X_5$.

(c) Compute the moment generating function of $Y = X_1 + 2X_2$.

2. Assume that X is chosen at random from numbers $-1, 0, 1$, each with equal probability.

(a) Compute the moment generating function of X.

(b) Let $X_1, X_2, \ldots$ be independent and all distributed as X, and let $S_n = X_1 + \ldots + X_n$. Show that, for every $\epsilon > 0$, $P(S_n \geq \epsilon n)$ and $P(S_n \leq -\epsilon n)$ are for large n smaller that n^{-10}.

(c) Let $X_1, X_2, \ldots$ be as in (b) and let M_n be the maximal absolute value of the sum of some n consecutive terms of $X_1, \ldots, X_{n^2}$. Show that $M_n/n \to 0$ in probability.

3. (I got this problem from a high-school student. This is a harder problem, and you do not have to turn it in.) The median of a sequence of $2n+1$ numbers is the element a of the sequence such n other elements are at least a and n other elements are at most a; that is, it is the middle number after the sequence is ordered. Roll a fair die $2n+1$ times and let M_n be the median of the numbers rolled. Approximate EM_n^2 for large n and find an upper bound for the error in your approximation. (*Hints*. The distribution of M_n is symmetric. With high probability, M_n is 3 or 4. Use Problem 4 in Chapter 10.)

Math 135B, Winter 2023.

Homework 2 Solutions.

1. Assume that $X_1, \dots, X_5$ are independent random variables with the same density

$$f(x) = \begin{cases} \frac{1}{e-1} e^x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the moment generating function of X_1.

Solution. By definition,

$$\phi_{X_1}(t) = \frac{1}{e-1} \int_0^1 e^x e^{tx}\, dx = \frac{e^{t+1}-1}{(e-1)(t+1)}.$$

(b) Compute the moment generating function of $S = X_1 + \ldots + X_5$.

Solution. By independence and (a),

$$\phi_S(t) = \phi_{X_1}(t)^5 = \left(\frac{e^{t+1}-1}{(e-1)(t+1)} \right)^5.$$

(c) Compute the moment generating function of $Y = X_1 + 2X_2$.

Solution. By independence and (a),

$$\phi_Y(t) = \phi_{X_1}(t)\phi_{2X_2}(t) = \phi_{X_1}(t)\phi_{X_2}(2t) = \left(\frac{e^{t+1}-1}{(e-1)(t+1)} \right) \left(\frac{e^{2t+1}-1}{(e-1)(2t+1)} \right).$$

2. Assume that X is chosen at random from numbers $-1, 0, 1$, each with equal probability.
(a) Compute the moment generating function of X.

Solution. By definition,

$$\phi_X(t) = \frac{1}{3}\left(e^{-t} + 1 + e^t\right).$$

(b) Let $X_1, X_2, \ldots$ be independent and all distributed as X, and let $S_n = X_1 + \ldots + X_n$. Show that, for every $\epsilon > 0$, $P(S_n \geq \epsilon n)$ and $P(S_n \leq -\epsilon n)$ are for large n smaller that n^{-10}.

Solution. Observe that $EX = 0$. By Theorem 10.2, $P(S_n \geq \epsilon n)$ is exponentially small, and therefore smaller than n^{-10}. By the remark after that theorem, the same is true for $P(S_n \leq -\epsilon n)$.

(c) Let $X_1, X_2, \ldots$ be as in (b) and let M_n be the maximal absolute value of the sum of some n consecutive terms of $X_1, \ldots, X_{n^2}$. Show that $M_n/n \to 0$ in probability.

Solution. There are less than n^2 sums of n consecutive terms, and each of these sums has the same distribution as S_n from (b). Therefore, as $M_n \geq 0$,

$$P(M_n \geq \epsilon n) \leq n^2 P(|S_n| \geq \epsilon n) = n^2(P(S_n \geq \epsilon n) + P(S_n \leq -\epsilon n)),$$

which goes to 0 by (b).

3. (I got this problem from a high-school student. This is a harder problem, and you do not have to turn it in.) The median of a sequence of $2n+1$ numbers is the element a of the sequence such n other elements are at least a and n other elements are at most a; that is, it is the middle number after the sequence is ordered. Roll a fair die $2n+1$ times and let M_n be the median of the numbers rolled. Approximate EM_n^2 for large n and find an upper bound for the error in your approximation. (*Hints*. The distribution of M_n is symmetric. With high probability, M_n is 3 or 4. Use Problem 4 in Chapter 10.)

Solution. Let $p_i = P(M_n = i)$, $i = 1, \ldots, 6$. By symmetry, $p_i = p_{7-i}$. Also, $p_1 + \ldots + p_6 = 1$, and so $p_3 = 1/2 - p_1 - p_2$. Then the expected value is

$$\begin{aligned} EM_n^2 &= p_1 + 4p_2 + 9p_3 + 16p_4 + 25p_5 + 36p_6 \\ &= 37p_1 + 29p_2 + 25p_3 \\ &= 12.5 + 12p_1 + 4p_2 \\ &= 12.5 + 8p_1 + 4(p_1 + p_2). \end{aligned}$$

Next we observe that

$$\begin{aligned} p_1 &= P(\text{no. of 1s} \geq n+1) \\ &= P(\text{Binomial}(2n+1, 1/6) \geq n+1) \\ &\leq P(\text{Binomial}(2n, 1/6) \geq n) \end{aligned}$$

and

$$\begin{aligned} p_1 + p_2 &= P(\text{combined no. of 1s and 2s} \geq n+1) \\ &= P(\text{Binomial}(2n+1, 1/3) \geq n+1) \\ &\leq P(\text{Binomial}(2n, 1/3) \geq n). \end{aligned}$$

By the answer to Problem 4, when $p = 1/3$ and $a = 1/2$,

$$I(a) = \frac{1}{2}\log\frac{3}{2} + \frac{1}{2}\log\frac{3}{4}$$

so that

$$P(\text{Binomial}(2n, 1/3) \geq n) \leq e^{-I(a)\cdot 2n} \leq e^{-0.1177\cdot n}.$$

Similarly, when $p = 1/6$ and $a = 1/2$,

$$I(a) = \frac{1}{2}\log 3 + \frac{1}{2}\log\frac{3}{5}$$

so that

$$P(\text{Binomial}(2n, 1/6) \geq n) \leq e^{-0.5877\cdot n}.$$

We conclude that

$$12.5 \leq EM_n^2 \leq 12.5 + 8e^{-0.5877\cdot n} + 4e^{-0.1177\cdot n}.$$

For $n \geq 38$, for example, EM_n^2 equals 12.5 to the first decimal.