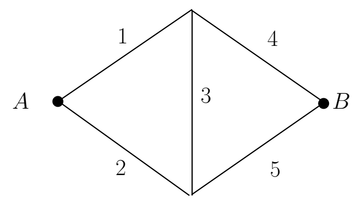
Math 135B, Winter 2023.

Homework 3

Due: Mar. 3, 2023

1. The road system between towns A and B is as in the picture. Assume that road i is open with probability p_i and that the roads are open independently. Compute the probability that A and B are connected.



2. In a sequence of n Bernoulli trials, each trial is independently a success with probability $p \in (0, 1)$ and failure with probability $1 - p$. Let p_n be the probability that there are no consecutive failures.

(a) Compute p_0, p_1, and p_2.

(b) Give the recursive equation, expressing p_n by p_{n-1} and p_{n-2}. Write down the general solution to the recursion and explain how you would determine the two constants. You do not need to compute the constants by hand.

(c) Compute $\lim_{n\to\infty}(p_{n+1}/p_n)$.

Here is a reminder on how to solve a second order linear recursion, given by $x_n = ax_{n-1} + bx_{n-2}$, where $a, b \in \mathbb{R}$. The characteristic equation $\lambda^2 = a\lambda + b$ either has two distinct solutions $\lambda_1 \neq \lambda_2$, or a single solution λ_1. In the first case, the general solution is $x_n = A\lambda_1^n + B\lambda_2^n$ and in the second case it is $x_n = \lambda_1^n(A + Bn)$. The constants A and B can be determined if we know, for example, x_0 and x_1, which gives two linear equations with two unknowns.

3. Assume that T is an exponential random variable with $ET = 1$, which is the time when a lightbulb goes out. Given $T = t$, the waiting time X for a repairman is (a) uniform on $[t/2, t]$, (b) exponential with expectation $1/t$. In each case, compute EX. In case (b), also compute the conditional density of T given $X = x$.

4. (A job interview problem from Google.) You can roll a 6-sided die at most 3 times. If you roll x on the first roll, you can either accept x dollars, or forgo this amount (so you get nothing from this roll) and continue rolling. If you continue, the same rule applies for the second roll. If you get to the third roll, you get x dollars if your roll x on the third roll and the game stops. What is the best strategy for playing this game and what is the expected payoff?