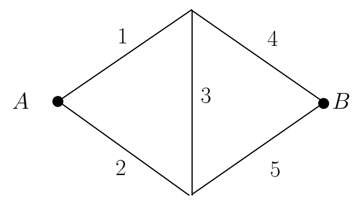
Math 135B, Winter 2023.

Homework 3

Due: Mar. 3, 2023

1. The road system between towns A and B is as in the picture. Assume that road i is open with probability p_i and that the roads are open independently. Compute the probability that A and B are connected.



2. In a sequence of n Bernoulli trials, each trial is independently a success with probability $p \in (0,1)$ and failure with probability $1-p$. Let p_n be the probability that there are no consecutive failures.

(a) Compute p_0, p_1, and p_2.

(b) Give the recursive equation, expressing p_n by p_{n-1} and p_{n-2}. Write down the general solution to the recursion and explain how you would determine the two constants. You do not need to compute the constants by hand.

(c) Compute $\lim_{n\to\infty}(p_{n+1}/p_n)$.

Here is a reminder on how to solve a second order linear recursion, given by $x_n = ax_{n-1} + bx_{n-2}$, where $a, b \in \mathbb{R}$. The characteristic equation $\lambda^2 = a\lambda + b$ either has two distinct solutions $\lambda_1 \neq \lambda_2$, or a single solution λ_1. In the first case, the general solution is $x_n = A\lambda_1^n + B\lambda_2^n$ and in the second case it is $x_n = \lambda_1^n(A + Bn)$. The constants A and B can be determined if we know, for example, x_0 and x_1, which gives two linear equations with two unknowns.

3. Assume that T is an exponential random variable with $ET = 1$, which is the time when a lightbulb goes out. Given $T = t$, the waiting time X for a repairman is (a) uniform on $[t/2, t]$, (b) exponential with expectation $1/t$. In each case, compute EX. In case (b), also compute the conditional density of T given $X = x$.

4. (A job interview problem from Google.) You can roll a 6-sided die at most 3 times. If you roll x on the first roll, you can either accept x dollars, or forgo this amount (so you get nothing from this roll) and continue rolling. If you continue, the same rule applies for the second roll. If you get to the third roll, you get x dollars if your roll x on the third roll and the game stops. What is the best strategy for playing this game and what is the expected payoff?

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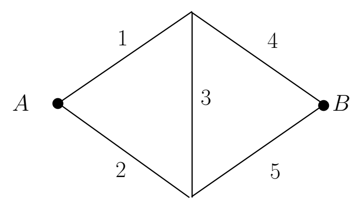
Homework 3 Solutions.

1. The road system between towns A and B is as in the picture. Assume that road i is open with probability p_i and that the roads are open independently. Compute the probability that A and B are connected.



Solution. We condition on the status of road 3. If road 3 is closed, then A and B are connected when either road 1 and road 4 are both open, or road 2 and road 5 are both open. If road 3 is closed, then A and B are connected when at least one of roads 1, 2 is open, and at least one of roads 4, 5 is open. This gives the answer

$$(1 - p_3)(p_1p_4 + p_2p_5 - p_1p_2p_4p_5) + p_3(p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5).$$

2. In a sequence of n Bernoulli trials, each trial is independently a success with probability $p \in (0, 1)$ and failure with probability $1 - p$. Let p_n be the probability that there are no consecutive failures.

(a) Compute p_0, p_1, and p_2.

(b) Give the recursive equation, expressing p_n by p_{n-1} and p_{n-2}. Write down the general solution to the recursion and explain how you would determine the two constants. You do not need to compute the constants by hand.

(c) Compute $\lim_{n\to\infty}(p_{n+1}/p_n)$.

Here is a reminder on how to solve a second order linear recursion, given by $x_n = ax_{n-1} + bx_{n-2}$, where $a, b \in \mathbb{R}$. The characteristic equation $\lambda^2 = a\lambda + b$ either has two distinct solutions $\lambda_1 \neq \lambda_2$, or a single solution λ_1. In the first case, the general solution is $x_n = A\lambda_1^n + B\lambda_2^n$ and in the second case it is $x_n = \lambda_1^n(A + Bn)$. The constants A and B can be determined if we know, for example, x_0 and x_1, which gives two linear equations with two unknowns.

Solution.

For (a), we have $p_0 = p_1 = 1$ and $p_2 = 1 - (1 - p)^2 = 2p - p^2$.

For (b), condition on the first trial. If it is a success, the remaining $n - 1$ trials must not have consecutive failures. If it is a failure, the next trial must be a success and then the remaining $n - 2$ trials must not have consecutive failures. Thus we have the recursive equation

$$p_n = p \cdot p_{n-1} + (1 - p)p \cdot p_{n-2},$$

with $p_0 = 1$, $p_1 = 1$. The characteristic equation

$$\lambda^2 - p\lambda - p(1 - p) = 0$$

has solutions

$$\lambda_{1,2} = \frac{p \pm \sqrt{p(4-3p)}}{2}.$$

So,

$$p_n = A \cdot \lambda_1^n + B \cdot \lambda_2^n.$$

Using $p_0 = p_1 = 1$, we get

$$\begin{aligned} 1 &= A + B \\ 1 &= A\lambda_1 + B\lambda_2 \end{aligned}$$

which are two linear equations with two unknowns, with a unique solution because $\lambda_1 \neq \lambda_2$.

For (c), observe that the root λ_2 (with the negative sign) is negative because

$$\sqrt{p(4-3p)} > \sqrt{p} > p$$

Therefore $A \neq 0$ as p_n cannot change sign. As also $|\lambda_2| < \lambda_1$, the limit is

$$\lambda_1 = \frac{p + \sqrt{p(4-3p)}}{2}.$$

For example, when $p = 1/2$,

$$\lambda_1 = \frac{1+\sqrt{5}}{4} \approx 0.809,$$

one half of the golden ratio.

3. Assume that T is an exponential random variable with $ET = 1$, which is the time when a lightbulb goes out. Given $T = t$, the waiting time X for a repairman is (a) uniform on $[t/2, t]$, (b) exponential with expectation $1/t$. In each case, compute ES. In case (b), also compute the conditional density of T given $X = x$.

Solution. In case (a),

$$EX = \int_0^\infty \frac{3t}{4} e^{-t}\, dt = \frac{3}{4}.$$

In case (b),

$$EX = \int_0^\infty \frac{1}{t} e^{-t}\, dt = \infty.$$

To compute the conditional density of T given $X = x$, we first compute the joint density of (T, X), then the marginal density of X, and finally use the formula.

We are given that the conditional density of X given $T = t$ is te^{-xt} and so the joint density of T and X is

$$f(t,x) = te^{-xt}e^{-t} = te^{-t(x+1)},$$

for $t, x \geq 0$. Then the density of X is

$$f_X(x) = \int_0^\infty te^{-t(x+1)}\, dt = \frac{1}{(x+1)^2},$$

for $x \geq 0$. The conditional density of T given $X = x$ is then

$$\frac{f(t,x)}{f_X(x)} = t(x+1)^2 e^{-t(x+1)}.$$

4. (A job interview problem from Google.) You can roll a 6-sided die at most 3 times. If you roll x on the first roll, you can either accept x dollars, or forgo this amount (so you get nothing from this roll) and continue rolling. If you continue, the same rule applies for the second roll. If you get to the third roll, you get x dollars if your roll x on the third roll and the game stops. What is the best strategy for playing this game and what is the expected payoff?

Solution. The key is to start with the final stage and work backwards. For this question, the final stage is after you have forgone the first two rolls. Then the payoff is the roll of a single die, with expectation 3.5. Now we go back one step. After the second roll, we choose either to have a third roll with an expected payoff of 3.5 or keep the current roll value. Clearly, we keep the roll value if it is larger than 3.5; in other words, if we roll 4, 5, or 6 on the second roll, we stop rolling. On the other hand, if we roll 1, 2, or 3 on the second roll, we roll the third time. Thus, the expected payoff before the second roll is

$$\frac{3}{6} \cdot 3.5 + \frac{1}{6} \cdot (4 + 5 + 6) = 4.25.$$

Now we go back one step further. After the first roll, we choose either to continue — which we already know has expected payoff 4.25 — or accept the current roll value. Now, we keep the current roll value if it is larger than 4.25, and continue otherwise. In other words, we continue exactly when we roll 5 or 6 on the first roll. Thus, the expected payoff before the first roll is

$$\frac{4}{6} \cdot 4.25 + \frac{1}{6} \cdot (5 + 6) = \frac{14}{3},$$

which is the expected payoff under our optimal strategy.