Math 135B, Winter 2023.

Homework 4

Due: Mar. 10, 2023

Note. You do not need to evaluate products of matrices by hand. Once you get your answer to the form in which the numerical answer can be obtained by a matrix computation, you may stop.

1. At each time $t = 0, 1, 2, \ldots$, each of the balls A and B is either in urn 1 or urn 2. At each time step, we select ball A with probability p and ball B with probability $1 - p$, and then keep the selected ball in the same urn with probability $1/3$ and place it in the other urn with probability $2/3$. Compute the probability that ball A is in urn 1 and ball B is in urn 2 after 6 time steps provided that: (a) initially, both balls are in urn 1; and (b) initially, each ball is independently put in one of the two urns with equal probability.

2. Assume that the Markov chain has states 1, 2, and 3 with the transition matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}.$$

Assume $P(X_0 = 1) = P(X_0 = 2) = 1/6$. Compute the following quantities.

(a) $P(X_0 = X_1 = \cdots = X_{10} = 3)$.

(b) EX_3 and EX_3^2.

(c) EX_3X_4.

3. Alice has two coins: coin 1 has probability 0.7 of Heads and coin 2 has probability 0.6 of Heads. She starts by tossing coin 1 twice. From then on, Alice tosses coin 1 unless her previous two tosses were both Heads, in which case she tosses coin 2. Bob knows Alice's procedure, and is in addition only told that Alice's 10th toss was Heads. Help Bob determine the (conditional) probability that Alice used coin 1 on the 10th toss.

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Homework 4 Solutions.

1. At each time $t = 0, 1, 2, \ldots$, each of the balls A and B is either in urn 1 or urn 2. At each time step, we select ball A with probability p and ball B with probability $1-p$, and then keep the selected ball in the same urn with probability $1/3$ and place it in the other urn with probability $2/3$. Compute the probability that ball A is in urn 1 and ball B is in urn 2 after 6 time steps provided that: (a) initially, both balls are in urn 1; and (b) initially, each ball is independently put in one of the two urns with equal probability.

Solution. We have a Markov chain with 4 states, which determine the positions of balls A and B, in order: 11, 12, 21, 22. The transition matrix is

$$P = \begin{bmatrix} 1/3 & 2(1-p)/3 & 2p/3 & 0 \\ 2(1-p)/3 & 1/3 & 0 & 2p/3 \\ 2p/3 & 0 & 1/3 & 2(1-p)/3 \\ 0 & 2p/3 & 2(1-p)/3 & 1/3 \end{bmatrix}$$

The two vectors of initial distribution are

$$\alpha_a = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$
$$\alpha_b = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

The answers to (a) and (b) are the second entries of $\alpha_a \cdot P^6$ and $\alpha_b \cdot P^6$, respectively.

2. Assume that the Markov chain has states 1, 2, and 3 with the transition matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}.$$

Assume $P(X_0 = 1) = P(X_0 = 2) = 1/6$. Compute the following quantities.

(a) $P(X_0 = X_1 = \cdots = X_{10} = 3)$.

Solution. As $P(X_0 = 3) = 2/3$ and $P_{33} = 1/6$, the answer is $2/(3 \cdot 6^{10})$.

(b) EX_3 and EX_3^2.

Solution. We have

$$EX_3 = \begin{bmatrix} P(X_3 = 1) & P(X_3 = 2) & P(X_3 = 3) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/6 & 2/3 \end{bmatrix} \cdot P^3 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and

$$EX_3^2 = \begin{bmatrix} P(X_3 = 1) & P(X_3 = 2) & P(X_3 = 3) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/6 & 2/3 \end{bmatrix} \cdot P^3 \cdot \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}.$$

(c) EX_3X_4.

Solution. Conditioning on the value of X_3,

$$\begin{aligned}&E(X_3X_4)\\&= P(X_3 = 1) \cdot E[X_4 \mid X_3 = 1] + P(X_3 = 2) \cdot 2E[X_4 \mid X_3 = 2] + P(X_3 = 3) \cdot 3E[X_4 \mid X_3 = 3].\end{aligned}$$

Also,

$$\begin{bmatrix} E[X_4 \mid X_3 = 1] \\ 2E[X_4 \mid X_3 = 2] \\ 3E[X_4 \mid X_3 = 3] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot P \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

and so

$$E(X_3X_4) = \begin{bmatrix} 1/6 & 1/6 & 2/3 \end{bmatrix} \cdot P^3 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot P \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

3. Alice has two coins: coin 1 has probability 0.7 of Heads and coin 2 has probability 0.6 of Heads. She starts by tossing coin 1 twice. From then on, Alice tosses coin 1 unless her previous two tosses were both Heads, in which case she tosses coin 2. Bob knows Alice's procedure, and is in addition only told that Alice's 10th toss was Heads. Help Bob determine the (conditional) probability that Alice used coin 1 on the 10th toss.

Solution. The Markov chain that keeps track of two successive outcomes of the tosses (current toss and previous toss) has states HH, HT, TH, TT and the transition matrix

$$P = \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.7 & 0.3 \\ 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.7 & 0.3 \end{bmatrix}.$$

The starting vector of probabilities is

$$\alpha = \begin{bmatrix} 0.7^2 & 0.7 \cdot 0.3 & 0.3 \cdot 0.7 & 0.3 \cdot 0.3 \end{bmatrix} = \begin{bmatrix} 0.49 & 0.21 & 0.21 & 0.09 \end{bmatrix}.$$

It is easier to compute the probability that coin 2 was used on the 10th toss. For this, we need to compute the conditional probability that, given the state at time 8 is either HH or TH, the state at time 7 is HH. The answer then is, by definition of conditional probability,

$$1 - \frac{(\alpha \cdot P^7)_1 \cdot 0.6}{(\alpha \cdot P^8)_1 + (\alpha \cdot P^8)_3}.$$