Math 135B, Winter 2023.

Homework 5

Due: Mar. 17, 2023

Note. Again, do not need to do matrix computations by hand. I recommend you use MATLAB; if you would prefer not to, stop when your answer to the form in which the numerical answer can be obtained by a matrix computation.

1. Solve Problem 1 in Chapter 13 of the lecture notes.

2. A casino is offering the following game. There are three six-sided dice, a red, a green, and a blue. Five sides of each die have symbols $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$, $+$, and $-$. The symbol on the sixth side depends on the die: on the red die it is $\mathcal{R}$ (so that the red die has two symbols $\mathcal{R}$); on the green die it is $\mathcal{B}$, and on the blue die it is $-$. If you roll $+$ or $-$, you respectively win or lose \$1 and the game is over. Otherwise, the symbol rolled determines the color of next die to roll ($\mathcal{R}$ for red, etc.). You continue rolling until your either roll $+$ or $-$.

(a) You are free to choose the die for your first roll. Determine which die you should choose. For this choice, compute your expected payoff and the expected number of die rolls.

(b) Now the casino changes the red die: the $-$ is replaced by $--$, and you lose \$2 if you roll $--$. Nothing else changes. Again, determine your optimal choice of the die, your expected payoff, and the expected number of die rolls in the game.

The way to approach this problem is as follows. Assume that you have a Markov chain whose first r states are absorbing and the other s states are transient. Then the transition matrix looks like this:

$$P = \begin{bmatrix} I_r & 0 \\ R & Q \end{bmatrix}.$$

Here, I_r is the $r \times r$ identity matrix, 0 is the $r \times s$ matrix of 0s, R is an $s \times r$ matrix, and Q is an $s \times s$ matrix. Suppose that S is the $s \times r$ matrix matrix whose entry S_{ij} is the probability that, starting from a transient state i, the chain ends up being absorbed into the absorbing state j. By conditioning on the first transition,

$$S_{ij} = R_{ij} + \sum_k Q_{ik} S_{kj},$$

where the sum is over all transient states k. (The above conditioning reflects that, in order to get to j, in a single step the chain either goes directly to j or to some transient state.) In the matrix form, we can write this $S = R + QS$, or

$$(I_s - Q)S = R.$$

Since we know that every entry of Q^n goes to 0 (by transience of the last s states), all eigenvalues of Q must be strictly less than 1 in absolute value. So, $I_s - Q$ must be invertible, and

$$S = (I_s - Q)^{-1} R.$$

Similarly, if N_{ij} is the expected total time, starting from a transient state i, that the process spends at a transient state j,

$$N_{ij} = \delta_{ij} + \sum_k Q_{ik} N_{kj}$$

(where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise), as the time spent at j is increased by 1 at the first step if $i = j$, but not if $i \neq j$. Letting N be the $s \times s$ matrix of these expected total times, we get $N = I_s + QN$, and so

$$N = (I_s - Q)^{-1}.$$

3. A random walker on $0, 1, 2, \ldots$ moves from each $i \geq 0$ either to $i + 1$ or to 0 (and nowhere else). Assume the move to $i + 1$ happens (1) with probability $1/2$; (2) with probability $e^{-2^{-i}}$. Let X_n be its position after n steps.

(a) Show that the Markov chain X_n is irreducible.

(b) Assume that the walker starts at 0. Let R be the first time the chain returns to 0 (so that R is at least 1). Compute $P(R > n)$.

(c) Compute f_0, the probability of return to 0 in each case.

(d) Determine recurrence or transience of all states.

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Homework 5 Solutions.

1. Solve Problem 1 in Chapter 13 of the lecture notes.

Solution. See the lecture notes.

2. A casino is offering the following game. There are three six-sided dice, a red, a green, and a blue. Five sides of each die have symbols $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$, $+$, and $-$. The symbol on the sixth side depends on the die: on the red die it is $\mathcal{R}$ (so that the red die has two symbols $\mathcal{R}$); on the green die it is $\mathcal{B}$, and on the blue die it is $-$. If you roll $+$ or $-$, you respectively win or lose \$1 and the game is over. Otherwise, the symbol rolled determines the color of next die to roll ($\mathcal{R}$ for red, etc.). You continue rolling until your either roll $+$ or $-$.

(a) You are free to choose the die for your first roll. Determine which die you should choose. For this choice, compute your expected payoff and the expected number of die rolls.

(b) Now the casino changes the red die: the $-$ is replaced by $--$, and you lose \$2 if you roll $--$. Nothing else changes. Again, determine your optimal choice of the die, your expected payoff, and the expected number of die rolls in the game.

The way to approach this problem is as follows. Assume that you have a Markov chain whose first r states are absorbing and the other s states are transient. Then the transition matrix looks like this:

$$P = \begin{bmatrix} I_r & 0 \\ R & Q \end{bmatrix}.$$

Here, I_r is the $r \times r$ identity matrix, 0 is the $r \times s$ matrix of 0s, R is an $s \times r$ matrix, and Q is an $s \times s$ matrix. Suppose that S is the $s \times r$ matrix matrix whose entry S_{ij} is the probability that, starting from a transient state i, the chain ends up being absorbed into the absorbing state j. By conditioning on the first transition,

$$S_{ij} = R_{ij} + \sum_k Q_{ik} S_{kj},$$

where the sum is over all transient states k. (The above conditioning reflects that, in order to get to j, in a single step the chain either goes directly to j or to some transient state.) In the matrix form, we can write this $S = R + QS$, or

$$(I_s - Q)S = R.$$

Since we know that every entry of Q^n goes to 0 (by transience of the last s states), all eigenvalues of Q must be strictly less than 1 in absolute value. So, $I_s - Q$ must be invertible, and

$$S = (I_s - Q)^{-1} R.$$

Similarly, if N_{ij} is the expected total time, starting from a transient state i, that the process spends at a transient state j,

$$N_{ij} = \delta_{ij} + \sum_k Q_{ik} N_{kj}$$

(where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise), as the time spent at j is increased by 1 at the first step if $i = j$, but not if $i \neq j$. Letting N be the $s \times s$ matrix of these expected total times, we get $N = I_s + QN$, and so

$$N = (I_s - Q)^{-1}.$$

Solution. In case (a), we have two transient states, and if we order states $+$, $-$, $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$:

$$R = \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}, Q = \begin{bmatrix} 1/3 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/3 \\ 1/6 & 1/6 & 1/6 \end{bmatrix},$$

and then

$$S = (I_3 - Q)^{-1} R = \begin{bmatrix} 36/79 & 43/79 \\ 35/79 & 44/79 \\ 30/79 & 49/79 \end{bmatrix}.$$

(Note that the rows sum to 1, as they must!) The vector of expected payoffs is

$$S \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7/79 \\ -9/79 \\ -19/79 \end{bmatrix}.$$

Starting with the red die is the best option, giving the average winning $-7/79 \approx -0.0886$. The expected number of rolls started from any of the three dice is given by the vector:

$$N \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (I - Q)^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 216/79 \\ 210/79 \\ 180/79 \end{bmatrix}.$$

As we are choosing the first die, the expected number of rolls is $216/79 \approx 2.7342$. (Note that this number has to be between 2 and 3!)

In case (b), Q is the same, but we have the additional transient state $--$, so

$$R = \begin{bmatrix} 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 0 \\ 1/6 & 1/3 & 0 \end{bmatrix},$$

and now

$$S = (I_3 - Q)^{-1} R = \begin{bmatrix} 36/79 & 20/79 & 23/79 \\ 35/79 & 37/79 & 7/79 \\ 30/79 & 43/79 & 6/79 \end{bmatrix},$$

and the vector of expected payoffs

$$S \cdot \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -30/79 \\ -16/79 \\ -25/79 \end{bmatrix}.$$

Now the green die is the optimal choice for the initial roll, with the payoff $-16/79 \approx -0.2025$. As Q is the same as in (a), the expected number of rolls is unchanged.

3. A random walker on $0, 1, 2, \ldots$ moves from each $i \geq 0$ either to $i + 1$ or to 0 (and nowhere else). Assume the move to $i + 1$ happens (1) with probability 1/2; (2) with probability $e^{-2^{-i}}$. Let X_n be its position after n steps.

(a) Show that the Markov chain X_n is irreducible.

Solution. Every state i communicates with 0, as the walk can get from i to 0 in 1 step and from 0 to i in i steps.

(b) Assume that the walker starts at 0. Let R be the first time the chain returns to 0 (so that R is at least 1). Compute $P(R > n)$.

Solution. The event $\{R > n\}$ happens exactly when none of the first n moves is to 0, which happens in case (1) with probability $1/2^n$ and in the case (2) with probability

$$\exp(-(1 + 2^{-1} + 2^{-2} + \ldots + 2^{-(n-1)})) = \exp(-2 + 2^{-n+1}).$$

(c) Compute f_0, the probability of return to 0 in each case.

Solution. Sending $n \to \infty$ in (b) we get in case (1)

$$1 - f_0 = \lim_{n\to\infty} P(T > n) = \lim_{n\to\infty} 1/2^n = 0,$$

and $f_0 = 1$. In case (2),

$$1 - f_0 = \lim_{n\to\infty} \exp(-2 + 2^{-n+1}) = e^{-2},$$

so that $f_0 = 1 - e^{-2}$.

(d) Determine recurrence or transience of all states.

Solution. As the chain is irreducible, by (a), the recurrence or transience of all states is determined by whether $f_0 = 1$ or $f_0 < 1$. Therefore, by (c), all states are recurrent in case (1) and transient in case (2).