Math 135B, Winter 2023.

Homework 6

Due: Feb. 24, 2023

1. Consider the branching process with the offspring distribution $p_0 = 3/32$, $p_1 = 15/32$, $p_2 = 9/32$, $p_3 = 5/32$. Let X_n be the population at generation n, with $X_0 = 1$.

(a) Find EX_5, the expectation and variance of the population at time 5.

(b) Find expected combined population of generations 0 through 5.

(c) Find the probability that the process ever dies out.

(d) Find the expression for the probability that the process dies by time 3 (i.e., at time 3 or earlier).

(e) Find $E(X_5 X_6)$.

2. Send-a-Dime, which begun in Colorado in 1935, was one of the first popular chain letters. A recipient received a list of 6 names and addresses, in order from top the bottom. Also, the recipient was instructed to send a dime to the to the top address, make a new list with the top name left off, add own name and address at the bottom, then make 5 copies of the letter with the new list and send them to 5 friends.
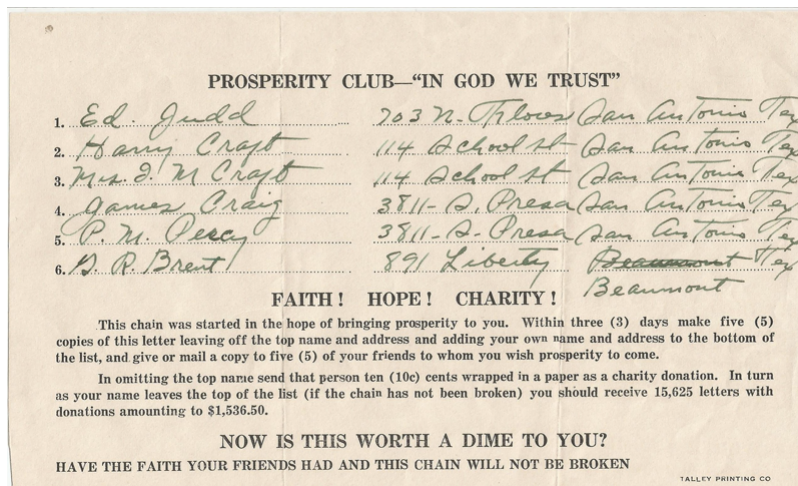
As usual, the chain letter contains hints about how unwise it is for the recipient to "break the chain," that is, to not follow the instructions and discard the letter. Send-a-Dime generated hundreds of thousands of letters within a year, causing all sorts of problems for the USPS. Federal law, specifically CFR §353.8, prohibits chain letters and related pyramid schemes.

Assume that each recipient independently follows the instructions with probability p, and otherwise discards the letter without sending anything.

(a) Assume $p = 1$. Verify that the recipient indeed eventually receives the sum advertised in the letter.

(b) Determine the expected amount of money you receive, depending on p, and also variance of this amount. Compute the variance when $p = 1/5$.

(c) Determine, using the computer if necessary, the probability that the chain letter dies out for $p = 1/2$, $p = 1/5$ and $p = 1/10$. For these values of p, determine also the probability that the recipient receives at least one dime back.

3. (This is a discrete version of the M/M/1 queue, from the book by Grinstead and Snell. You do not need to turn in the solution to this problem.) Each minute, either 1 or 0 customers arrive to a server, based on a toss of the *arrival* coin with Heads probability p. Upon arrival, the customer either starts being served or joins the queue. When a customer is being served, the service is complete in that minute based on the independent *service* coin with heads probability r.

For example, if at some minute the queue is of size 4 and both coins come out Heads, then the queue is unchanged next time, but if only the service coin comes out Heads, the queue is reduced to 3. If the queue is empty, and a customer is being served, and both coins come out Heads, the situation is again the same the next minute, but now if only the service coin comes out Heads, the next minute nobody is served and the server has a free minute. In the latter situation, the service coin is immaterial, but if the arrival coin comes out Heads, in the next minute the queue is empty and a customer is being served. Note that every customer is served at least one minute.

At the start of the process (minute 0), there is a customer being served and an empty queue. Determine the probability that the server ever has a free minute.

(*Hints.* The service time is Geometric. Consider a branching process in which the offspring of a customer are the new customers that arrive during the service time. The generating function of the offspring distribution can be obtained by conditioning. Observe that the free minute happens if any only if the branching process dies out.)