Math 135B, Winter 2023.

Homework 6

Due: Feb. 24, 2023

1. Consider the branching process with the offspring distribution $p_0 = 3/32$, $p_1 = 15/32$, $p_2 = 9/32$, $p_3 = 5/32$. Let X_n be the population at generation n, with $X_0 = 1$.

(a) Find EX_5, the expectation and variance of the population at time 5.

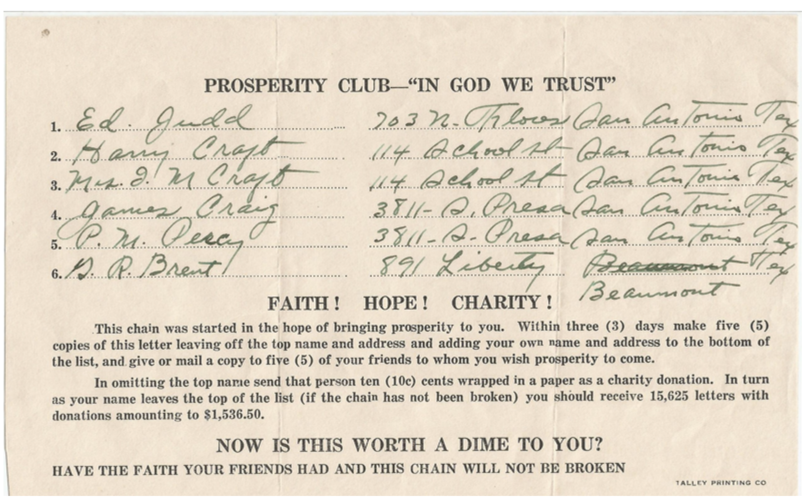
(b) Find expected combined population of generations 0 through 5.

(c) Find the probability that the process ever dies out.

(d) Find the expression for the probability that the process dies by time 3 (i.e., at time 3 or earlier).

(e) Find $E(X_5X_6)$.

2. Send-a-Dime, which begun in Colorado in 1935, was one of the first popular chain letters. A recipient received a list of 6 names and addresses, in order from top the bottom. Also, the recipient was instructed to send a dime to the to the top address, make a new list with the top name left off, add own name and address at the bottom, then make 5 copies of the letter with the new list and send them to 5 friends.



PROSPERITY CLUB—"IN GOD WE TRUST"

1. Ed. Judd 703 N. Flores San Antonio Tex
2. Harry Craft 114 School St San Antonio Tex
3. Mrs. J. M Craft 114 School St San Antonio Tex
4. James Craig 3811- S. Presa San Antonio Tex
5. P. M. Percy 3811- S. Presa San Antonio Tex
6. D. R. Brent 891 Liberty ~~Beaumont~~ Tex Beaumont

FAITH! HOPE! CHARITY!

This chain was started in the hope of bringing prosperity to you. Within three (3) days make five (5) copies of this letter leaving off the top name and address and adding your own name and address to the bottom of the list, and give or mail a copy to five (5) of your friends to whom you wish prosperity to come.

In omitting the top name send that person ten (10c) cents wrapped in a paper as a charity donation. In turn as your name leaves the top of the list (if the chain has not been broken) you should receive 15,625 letters with donations amounting to $1,536.50.

NOW IS THIS WORTH A DIME TO YOU?

HAVE THE FAITH YOUR FRIENDS HAD AND THIS CHAIN WILL NOT BE BROKEN

TALLEY PRINTING CO

As usual, the chain letter contains hints about how unwise it is for the recipient to "break the chain," that is, to not follow the instructions and discard the letter. Send-a-Dime generated hundreds of thousands of letters within a year, causing all sorts of problems for the USPS. Federal law, specifically CFR §353.8, prohibits chain letters and related pyramid schemes.

Assume that each recipient independently follows the instructions with probability p, and otherwise discards the letter without sending anything.

(a) Assume $p = 1$. Verify that the recipient indeed eventually receives the sum advertised in the letter.

(b) Determine the expected amount of money you receive, depending on p, and also variance of this amount. Compute the variance when $p = 1/5$.

(c) Determine, using the computer if necessary, the probability that the chain letter dies out for $p = 1/2$, $p = 1/5$ and $p = 1/10$. For these values of p, determine also the probability that the recipient receives at least one dime back.

3. (This is a discrete version of the M/M/1 queue, from the book by Grinstead and Snell. You do not need to turn in the solution to this problem.) Each minute, either 1 or 0 customers arrive to a server, based on a toss of the *arrival* coin with Heads probability p. Upon arrival, the customer either starts being served or joins the queue. When a customer is being served, the service is complete in that minute based on the independent *service* coin with heads probability r.

For example, if at some minute the queue is of size 4 and both coins come out Heads, then the queue is unchanged next time, but if only the service coin comes out Heads, the queue is reduced to 3. If the queue is empty, and a customer is being served, and both coins come out Heads, the situation is again the same the next minute, but now if only the service coin comes out Heads, the next minute nobody is served and the server has a free minute. In the latter situation, the service coin is immaterial, but if the arrival coin comes out Heads, in the next minute the queue is empty and a customer is being served. Note that every customer is served at least one minute.

At the start of the process (minute 0), there is a customer being served and an empty queue. Determine the probability that the server ever has a free minute.

(*Hints*. The service time is Geometric. Consider a branching process in which the offspring of a customer are the new customers that arrive during the service time. The generating function of the offspring distribution can be obtained by conditioning. Observe that the free minute happens if any only if the branching process dies out.)

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Homework 6 Solutions.

1. Consider the branching process with the offspring distribution $p_0 = 3/32$, $p_1 = 15/32$, $p_2 = 9/32$, $p_3 = 5/32$. Let X_n be the population at generation n, with $X_0 = 1$.

(a) Find EX_5, the expectation and variance of the population at time 5.

Solution.

We get $\mu = EX_1 = 3/2$ and so $EX_5 = \mu^5 = (3/2)^5 = 243/32$. Also, $EX_1^2 = 3$, so $\sigma^2 = \mathrm{Var}(X_1) = 3 - 9/4 = 3/4$, so that

$$\mathrm{Var}(X_5) = \frac{\sigma^2\mu^5(1-\mu^5)}{\mu(1-\mu)} = \frac{3^5 \cdot 211}{2^{10}} \approx 50.0713.$$

(b) Find expected combined population of generations 0 through 5.

Solution. The combined population is $X_0 + \ldots + X_5$, with expectation

$$1 + (3/2) + (3/2)^2 + \ldots + (3/2)^5 = 2 \cdot ((3/2)^6 - 1) = 665/32 \approx 20.78.$$

(c) Find the probability that the process ever dies out.

Solution. We have

$$\phi(s) = \frac{3}{32} + \frac{15}{32}s + \frac{9}{32}s^2 + \frac{5}{32}s^3.$$

The equation $\phi(s) = s$ reduces to $5s^3 + 9s^2 - 17s + 3 = 0$, or

$$(s-1)(s+3)(5s-1) = 0,$$

with the smallest positive solution $s = 1/5$, which is the answer.

(d) Find the expression for the probability that the process dies by time 3 (i.e., at time 3 or earlier).

Solution. This is

$$\delta_3 = P(X_3 = 0) = \phi(\phi(\phi(0))) \approx 0.1655.$$

(e) Find $E(X_5X_6)$.

Solution. Conditioning on X_5 gives

$$
\begin{aligned}
E(X_5X_6) &= \sum_{k\geq 0} E(X_5X_6 \mid X_5 = k)\cdot P(X_5 = k) \\
&= \sum_{k\geq 0} k\cdot E(X_6 \mid X_5 = k)\cdot P(X_5 = k) \\
&= \sum_{k\geq 0} k\cdot \frac{3}{2}\cdot k\cdot P(X_5 = k) \\
&= \frac{3}{2}\cdot \sum_{k\geq 0} k^2\cdot P(X_5 = k) \\
&= \frac{3}{2}\cdot EX_5^2 = \frac{3}{2}\cdot (\mathrm{Var}(X_5) + (EX_5)^2) = \frac{3}{2}\cdot\left(\frac{3^5\cdot 211}{2^{10}} + (3/2)^{10}\right).
\end{aligned}
$$

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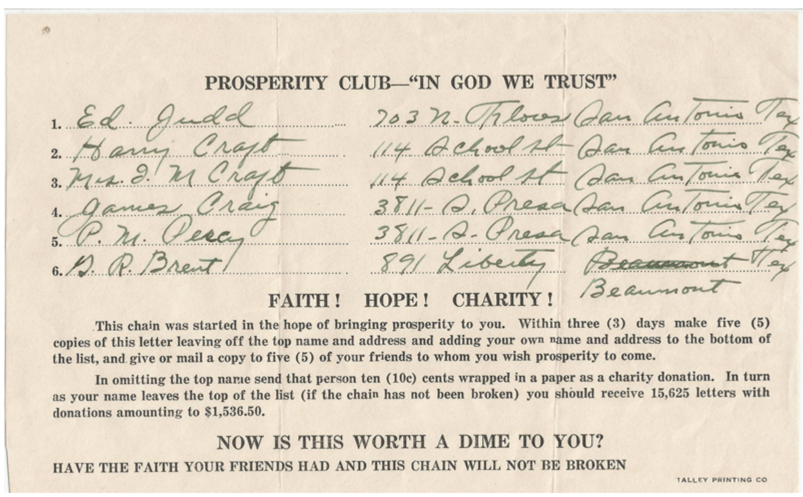
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Assume that each recipient independently follows the instructions with probability p, and otherwise discards the letter without sending anything.

(a) Assume $p = 1$. Verify that the recipient indeed eventually receives the sum advertised in the letter.

Solution. This amount is $5^6/10$ dollars.

(b) Determine the expected amount of money you receive, depending on p, and also variance of this amount. Compute the variance when $p = 1/5$.

Solution. We have a branching process with offspring distribution $p_0 = 1 - p$ and $p_5 = p$. So, $\mu = 5p$, and $EX_7 = (5p)^7$. Also $\sigma^2 = 25p - (5p)^2$ and we can use the formula

$$\mathrm{Var}(X_7) = \frac{\sigma^2 \mu^7 (1 - \mu^7)}{\mu(1 - \mu)},$$

when $p \neq 1/5$. When $p = 1/5$, $\sigma^2 = 4$ and

$$\mathrm{Var}(X_7) = 7\sigma^2 = 28.$$

The amount S of money (in dimes) received is exactly $S = X_7/5$, so $ES = (5p)^7/5$ and $\mathrm{Var}(S) = \mathrm{Var}(X_7)/25$. When $p = 1/5$, $ES = 1/5$ and $\mathrm{Var}(S) = 28/25$.

(c) Determine, using the computer if necessary, the probability that the chain letter dies out for $p = 1/2$, $p = 1/5$ and $p = 1/10$. For these values of p, determine also the probability that the recipient receives at least one dime back.

Solution. We have $\phi(s) = 1 - p + ps^5$. When $p \leq 1/5$, the chain letter dies out with probability 1. When $p > 1/5$, we need to find the smallest positive solution of $\phi(s) = s$. For $p = 1/2$, this solution is about 0.518790.

The probability that the recipient receives at least one dime back is the probability that the chain does not die by generation 7, so it equals

$$P(X_7 > 0) = 1 - P(X_7 = 0) = 1 - \phi_{X_7}(0).$$

So we need to compute the 7th iterate of ϕ at 0. When $p = 1/2$, $\phi_{X_7}(0) \approx 0.518789$, so it barely differs from the limit. The answer then is $P(X_7 > 0) \approx 0.481211$. When $p = 1/5$, $P(X_7 > 0) \approx 0.054387$, and when $p = 1/10$, $P(X_7 > 0) \approx 0.001100$.

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(*Hints*. The service time is Geometric. Consider a branching process in which the offspring of a customer are the new customers that arrive during the service time. The generating function of the offspring distribution can be obtained by conditioning. Observe that the free minute happens if any only if the branching process dies out.)

Solution. Proceeding as in the hint, assume that X has the offspring distribution and that N has the service time distribution. Thus, N is Geometric(r). Conditioned on $N = n$, X is Binomial(n, p), as the sum of independent indicators, each of which is Bernoulli(p). Therefore,

$$\begin{aligned}\phi(s) = Es^X &= \sum_{n=1}^{\infty} E[s^X \mid N = n] \cdot P(N = n) \\ &= \sum_{n=1}^{\infty}(ps + 1 - p)^n \cdot (1 - r)^{n-1} r \\ &= r(ps + 1 - p) \sum_{n=1}^{\infty}((1 - r)(ps + 1 - p))^{n-1} \\ &= \frac{r(ps + 1 - p)}{1 - ((1 - r)(ps + 1 - p))} \\ &= \frac{srp + r(1 - p)}{-sp(1 - r) + p + r - rp}\end{aligned}$$

Also, we know that $EX = EN \cdot p = p/r$.

The free minute will happen exactly when the branching process dies out. This will happen with probability 1 if $EX \leq 1$, that is, when $p \leq r$. If $p > r$, it will happen with probability that is the the smallest positive solution to the equation $\phi(s) = s$, which is the quadratic equation

$$s^2 p(1 - r) - s(p + r - 2rp) + r(1 - p) = 0,$$

with one solution $s = 1$ and the other

$$s = \frac{r(1 - p)}{p(1 - r)},$$

which is the answer when $p > r$.