Math 135B, Winter 2023.

Homework 8

Due: Mar. 10, 2023

Note. These three problems are adapted from job interviews on Wall Street. Assume that you may use a computer during the interview for coding (but not for looking for a solution on the internet). These seem difficult, albeit not impossible, problems to solve completely under pressure, so I suspect that the interviewers mainly want to gauge the candidate's reaction in such a situation. To get a bit of experience on how you might react, you are required, as a part of your submitted work, to write down an answer to each of the three problems *before you see the solution*. Each of your answers needs to at least outline an approach in a coherent language, and needs to contain a numerical answer, which can be just a guess if you are unable to make much progress; imagine that you are addressing your response to an interviewer. After that, you may verify and complete your work by consulting the solutions. (Of course, there is no penalty for incorrect preliminary answers.)

1. Alice has a coin with probability 0.6 of Heads, and Bob has coin with probability 0.3 of Heads. They toss a coin repeatedly: if the number of Heads tossed so far is odd, Alice tosses her coin, while if the number of Heads tossed so far is even, Bob tosses his coin. Let p_n be the probability of even number of Heads after n tosses. Approximate p_{1000}.

2. Two tokens are both initially positioned at 0 and move on nonnegative integers $0, 1, \ldots$ as follows. There are two coins: the red coin is fair, with Heads probability $1/2$, while the blue coin has Heads probability $2/3$. Each minute, exactly one of the token makes a move based on the outcome of a toss of one of the coins: if the toss is Heads, the move is two unit steps to the right, and if the toss is Tails, the move is one unit step to the right. Here are the rules on which coin is used and which token makes the move:

- if both token occupy the same position, the red coin is used for the toss, and one of the tokens makes the move and the other stays put;
- otherwise, the blue coin is used for the toss, and the token that is behind (i.e., to the left) makes the move and the token that is ahead stays put.

Let p_n be the probability that the two tokens ever both simultaneously occupy n. Approximate p_{1000}.

3. A casino offers the following card game. A standard deck is shuffled and the dealer draws cards one by one, without replacement. You may ask the dealer to stop at any time. For each red card drawn, you win \$1; and for each black card drawn, you lose \$1. How much are you willing to pay to play the game?

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Homework 8 Solutions.

1. Alice has a coin with probability 0.6 of Heads, and Bob has coin with probability 0.3 of Heads. They toss a coin repeatedly: if the number of Heads tossed so far is odd, Alice tosses her coin, while if the number of Heads tossed so far is even, Bob tosses his coin. Let p_n be the probability of even number of Heads after n tosses. Approximate p_{1000}.

Solution. While it is possible to solve this directly by recursion, we can interpret this process as a two state Markov chain X_n, where states are 0 and 1 and represent the number of Heads modulo 2. Clearly, $X_0 = 0$. Further, the chain has transition matrix

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}.$$

The probability in question exactly equals P_{00}^{1000}, but we can approximate it by the first entry of the invariant distribution π_0. We can compute by hand $\pi_0 = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$. So the answer is 2/3.

2. Two tokens are both initially positioned at 0 and move on nonnegative integers $0, 1, \ldots$ as follows. There are two coins: the red coin is fair, with Heads probability 1/2, while the blue coin has Heads probability 2/3. Each minute, exactly one of the token makes a *move* based on the outcome of a toss of one of the coins: if the toss is Heads, the move is two unit steps to the right, and if the toss is Tails, the move is one unit step to the right. Here are the rules on which coin is used and which token makes the move:

- if both token occupy the same position, the red coin is used for the toss, and one of the tokens makes the move and the other stays put;
- otherwise, the blue coin is used for the toss, and the token that is behind (i.e., to the left) makes the move and the token that is ahead stays put.

Let p_n be the probability that the two tokens ever both simultaneously occupy n. Approximate p_{1000}.

Solution. Let p_i^n be the probability in question if one of the tokens starts at 0 and the other at i, $i = 0, 1, 2$. Clearly $p_i^n = 0$ if $n < i$ for all i and $n \in \mathbb{Z}$, and $p_0^0 = 1$. Also, by conditioning on the first move, for all n,

$$\begin{aligned} p_0^n &= \frac{1}{2} p_1^n + \frac{1}{2} p_2^n + \delta_{0n} \\ p_1^n &= \frac{1}{3} p_0^{n-1} + \frac{2}{3} p_1^{n-1} \\ p_2^n &= \frac{1}{3} p_1^{n-1} + \frac{2}{3} p_0^{n-2} \end{aligned}$$

Here, $\delta_{xy} = 1$ when $x = y$ and 0 otherwise. This is already enough to compute the probabilities for, say, $n = 40$, and to conjecture that they all converge rapidly to 6/17.

To proceed with a proof, we first substitute

$$p_0^{n-1} = \frac{1}{2}p_1^{n-1} + \frac{1}{2}p_2^{n-1} + \delta_{1n}$$
$$p_0^{n-2} = \frac{1}{2}p_1^{n-2} + \frac{1}{2}p_2^{n-2} + \delta_{2n}$$

into the other two equations, to get

$$p_1^n = \frac{5}{6}p_1^{n-1} + \frac{1}{6}p_2^{n-1} + \frac{1}{3}\delta_{1n}$$
$$p_2^n = \frac{1}{3}p_1^{n-1} + \frac{1}{3}p_1^{n-2} + \frac{1}{3}p_2^{n-2} + \frac{2}{3}\delta_{2n}$$

If we let $q_1^n = p_1^{n-1}$ and $q_2^n = p_2^{n-1}$, we get the linear recursion

$$\begin{bmatrix} p_1^n \\ p_2^n \\ q_1^n \\ q_2^n \end{bmatrix} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1^{n-1} \\ p_2^{n-1} \\ q_1^{n-1} \\ q_2^{n-1} \end{bmatrix} + \begin{bmatrix} \delta_{1n}/3 \\ 2\delta_{2n}/3 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} p_1^0 \\ p_2^0 \\ q_1^0 \\ q_2^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Call the above 4×4 matrix P. This is a stochastic matrix, a transition matrix of an irreducible and aperiodic Markov chain, with invariant distribution

$$\pi_0 = \left[12/17, 3/17, 1/17, 1/17\right].$$

By the recursion, we can compute

$$\begin{bmatrix} p_1^2 \\ p_2^2 \\ q_1^2 \\ q_2^2 \end{bmatrix} = \begin{bmatrix} 5/18 \\ 7/9 \\ 1/3 \\ 0 \end{bmatrix},$$

and then for $n \geq 2$

$$\begin{bmatrix} p_1^n \\ p_2^n \\ q_1^n \\ q_2^n \end{bmatrix} = P^{n-2} \begin{bmatrix} 5/18 \\ 7/9 \\ 1/3 \\ 0 \end{bmatrix}.$$

We know that as $n \to \infty$, P^n converges to the matrix with all rows equal to π_0, and so all probabilities converge to

$$\pi_0 \cdot \begin{bmatrix} 5/18 \\ 7/9 \\ 1/3 \\ 0 \end{bmatrix} = 6/17,$$

as predicted.

Note. When the same coin is always used, there is a much shorter and slicker solution, that also works for all distributions of steps. Suppose that the fair red coin is always used. Imagine that one of the tokens is moved by Alice and the other by Bob, and that Alice's token is the one moved when they occupy the same position. We claim the following:

(1) whenever Alice moves, she makes a step of size 1 or 2 with equal probability and the same is true for Bob;

(2) Alice's and Bob's steps are independent; and

(3) if they both visit n, they *must* be there at the same time.

Parts (1) and (2) are clear. To verify part (3), observe that if they both reach n and, say, Alice is at n first, then she must wait there for Bob to reach it. The probability that Alice visits n is 1 over the expected size of her step, thus by (1) it equals $1/(3/2) = 2/3$. By (3), we need to compute the probability of the event $\{\text{Alice visits } n\} \cap \{\text{Bob visits } n\}$. By (2), this probability is $(2/3)^2 = 4/9$.

3. A casino offers the following card game. A standard deck is shuffled and the dealer draws cards one by one, without replacement. You may ask the dealer to stop at any time. For each red card drawn, you win \$1; and for each black card drawn, you lose \$1. How much are you willing to pay to play the game?

Solution. We will solve this problem by computing the expected payoff $f(b, r)$ for arbitrary deck of $b + r$ cards, b black ones and r red ones, under optimal strategy. Clearly, $f(0, r) = r$ for any $r \geq 0$, and $f(b, 0) = 0$ for any $b \geq 0$. Otherwise, we compute the expected payoff provided we decide to play the first card. If it is strictly positive, we do play the first card, otherwise we quit immediately. If we decide to play, the probability that the first card is black is $b/(b + r)$, in which case we lose a dollar and the numbers change to $(b - 1, r)$, and the probability that the next card is red is $r/(b + r)$, in which case we win a dollar and the numbers change to $(b, r - 1)$. This yields the recursion:

$$f(b, r) = \max\{0, \frac{b}{b+r}(-1 + f(b-1, r)) + \frac{r}{b+r}(1 + f(b, r-1))\}.$$

Using this recursion, we can calculate $f(b, r)$ for all pairs (b, r). The answer is $f(26, 26)$, which is the expected payoff at the beginning of the game under optimal strategy. A computer calculation gives $f(26, 26) \approx 2.6245$.