Math 135B, Winter 2023.
Feb. 3, 2023.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

$$\mathrm{Var}\left(\sum_{i=1}^{N} X_i\right) = \mathrm{Var}(X_1)\cdot EN + (EX_1)^2\cdot \mathrm{Var}(N)$$

1. Assume $X_1, X_2, X_3, \ldots$ are your daily dollar winnings playing a certain game at a casino on days $1, 2, 3, \ldots$. The winnings X_i are independent and they all have values 1, 2, or 3, with the following probabilities: $P(X_i = 1) = 1/2$, $P(X_i = 2) = 1/3$, $P(X_i = 3) = 1/6$. (This sequence of random variables will appear again in Problems 2 and 4(a).)

Let N_n be the number of integers $i \in [1, n]$ such that $X_i = 1$ and $X_{i+1} = 2$, that is, the number of days, among the first n, such that you win 1 on that day, but you win 2 on the following day.

(a) Compute EN_n.

Let $I_i = I\{X_i = 1, X_{i+1} = 2\}$. Then $EI_i = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

$N_n = \sum_{i=1}^{n} I_i$, so $EN_n = \sum_{i=1}^{n} EI_i = \underline{\underline{\frac{n}{6}}}$

(b) Compute $\mathrm{Var}(N_n)$.

If $j \geq i+2$, I_i and I_j are independent, so that $\mathrm{Cov}(I_i, I_j) = 0$.

$$\mathrm{Var}(N_n) = \sum_{i=1}^{n} \mathrm{Var}(I_i) + 2 \sum_{i<j} \mathrm{Cov}(I_i, I_j)$$

$$= \sum_{i=1}^{n} (EI_i - (EI_i)^2) + 2 \sum_{i=1}^{n-1} \mathrm{Cov}(I_i, I_{i+1})$$

$$= n\left(\frac{1}{6} - \frac{1}{36}\right) + 2(n-1)\left(\underbrace{E(I_i I_{i+1})}_{0} - \frac{1}{36}\right)$$

0, as $I_i I_{i+1} = 0$

$$= \frac{5n - 2(n-1)}{36} = \frac{3n+2}{36}$$

(c) Determine, with proof, the number c so that $\frac{N_n}{n} \to c$ in probability, as $n \to \infty$.

$Y_n = \frac{N_n}{n}$. Then $EY_n = \frac{1}{6}$ and $\mathrm{Var}(Y_n) = \frac{1}{n^2} \cdot \mathrm{Var}(N_n)$

$= \frac{3n+2}{36 n^2} \to 0$,

Therefore $\underline{\underline{Y_n \to \frac{1}{6}}}$ in probability.

2. Assume $X_1, X_2, X_3, \ldots$ are your daily winnings in problem 1, that is, X_i are independent and they all have values 1, 2, or 3, with the following probabilities: $P(X_i = 1) = 1/2$, $P(X_i = 2) = 1/3$, $P(X_i = 3) = 1/6$. Let $S_n = X_1 + X_2 + \ldots + X_n$ be the combined amount you win in the first n days.

(a) Compute the moment generating function of X_1.

$$\varphi_{X_1}(t) = \frac{1}{2}e^{t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}.$$

(b) Compute the moment generating function of $S_3 = X_1 + X_2 + X_3$.

$$\varphi_{S_3}(t) = \varphi_{X_1}(t)^3 = \left(\frac{1}{2}e^{t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}\right)^3$$

(c) Explain how you would get a good upper bound on $P(S_n \geq 2n)$, but do not carry out the computation. Is the probability $P(S_n \geq 2n)$ smaller than n^{-5} for large n?

$$\mu = EX_1 = \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{6}\cdot 3 = \frac{5}{2} < 2$$

Therefore,

$$I(2) = \sup\{t>0 : 2t - \log \varphi_{X_1}(t)\} > 0$$

(In fact, $I(2)$ is a bit larger that 0.0935)

then $P(S_n \geq 2n) \leq e^{-I(2)n}$, which is much smaller than n^{-5} for large n.

3. Assume the random variable X is uniform on $[0,1]$. Given that $X = x$, the random variable T is exponential with expectation $1/x$ (that is, $f_T(t \mid X = x) = xe^{-xt}$ for $t \geq 0$).

(a) Compute ET.

$$\underline{\underline{ET}} = \int_0^1 E[T \mid X=x]\, dx = \int_0^1 \frac{1}{x}\, dx = \underline{\underline{\infty}}$$

(b) Compute the joint density $f(x,t)$ of X and T.

$$f(x,t) = \begin{cases} x e^{-xt} & x \in [0,1],\ t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(c) Compute the density of $f_T(t)$ of T. (You may use: $\int xe^{-tx}\, dx = -\frac{x}{t}e^{-tx} - \frac{1}{t^2}e^{-tx} + C$.)

$$f_T(t) = \int_0^1 x e^{-xt}\, dx$$

$$= -\frac{x}{t} e^{-tx} - \frac{1}{t^2} e^{-tx} \Big|_{x=0}^{x=1}$$

$$= -\frac{1}{t} e^{-t} - \frac{1}{t^2} e^{-t} + \frac{1}{t^2}$$

$$= \frac{1-(t+1)e^{-t}}{t^2} \quad \text{for } t \geq 0$$

(0 otherwise)

(d) Compute the conditional density of X given $T = t$. $= \frac{f(x,t)}{f_T(t)}$

$$= \frac{x t^2 e^{-xt}}{1-(t+1)e^{-t}} \quad \text{for } x \in [0,1]$$

(0 otherwise)

4. (a) Assume $X_1, X_2, X_3, \ldots$ are your daily winnings in problem 1, that is, X_i are independent and they all have values 1, 2, or 3, with the following probabilities: $P(X_i = 1) = 1/2$, $P(X_i = 2) = 1/3$, $P(X_i = 3) = 1/6$.

Assume now that each day you also roll a fair die, which does not contribute to your winnings, but decides when the game ends: the day you roll a 1 or a 2 is the last day you play the game. Let W be the combined amount you win from all days you play, including the last. (For example, if your rolls on first two days are 4 and 1, and $X_1 = 3$, $X_2 = 2$, then $W = 5$.) Compute EW and $\mathrm{Var}(W)$. (You may use that geometric random variable with parameter p has expectation $1/p$ and variance $\frac{1-p}{p^2}$.)

N = no. of days you roll is Geometric$(1/3)$.

$EN = 3$, $\quad \mathrm{Var}(N) = \dfrac{1 - 1/3}{1/9} = 6$

$EX_1 = \frac{5}{3}$ $\quad \mathrm{Var}(X_1) = EX_1^2 - \left(\frac{5}{3}\right)^2 = \frac{1}{2} + \frac{4}{3} + \frac{9}{6} - \left(\frac{5}{3}\right)^2$

$= \dfrac{30 - 25}{9} = \dfrac{5}{9}$

$\underline{EW} = EN \cdot EX_1 = \underline{5}$

$\underline{\mathrm{Var}(W)} = \mathrm{Var}(X_1) \cdot EN + (EX_1)^2 \cdot \mathrm{Var}(N)$

$= \frac{5}{9} \cdot 3 + \left(\frac{5}{3}\right)^2 \cdot 6 = \underline{\frac{55}{3}}$

(b) Now the casino chooses a random number U, which is chosen before the games begin, and then remains unchanged. The number U is uniform in $[0, 1/2]$. Given $U = u$, your daily winnings are still independent, still have values 1, 2, or 3, but now the probabilities are: $P(X_i = 1) = 1/2$. $P(X_i = 2) = u$, $P(X_i = 3) = 1/2 - u$. Assume you play the game for exactly 8 days (so, unlike in (a), there is no die) and W is again ~~[illegible]~~ the combined amount you win. Compute EW.

$E[X_1 \mid U = u] = \frac{1}{2} + 2u + 3\left(\frac{1}{2} - u\right) = 2 - u,$

so $E[W \mid U = u] = 8(2 - u)$.

$\underline{E[W]} = \int_0^{1/2} E[W \mid U = u]\, 2 \cdot du = 16 \int_0^{1/2} (2 - u)\, du$

$= 16 \cdot \left(1 - \frac{1}{8}\right) = \underline{14}$