Math 135B, Winter 2023.
Mar. 3, 2023.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices. books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Alice and Bob each toss a coin once per day on days $0, 1, 2 \ldots$. Each day, each of them tosses either the red or the blue coin. The red coin has Heads probability 0.6, while the blue coin Heads probability 0.2. On day 0, both Alice and Bob toss the red coin. Subsequently, on day n, $n = 1, 2, \ldots$: Bob tosses the coin of the same color as Alice tossed on day $n-1$; and Alice tosses the red coin if Bob tossed Heads on day $n-1$, and she tosses the blue coin otherwise.

(a) Determine the transition probability matrix for the Markov chain on states $1, 2, 3, 4$, which keeps track, in order, the colors of Alice's and Bob's coins each day: RR, RB, BR, BB (R=red, B=blue). Is the chain irreducible? Aperiodic?

$$P = \begin{array}{c|cccc} & \text{1 RR} & \text{2 RB} & \text{3 BR} & \text{4 BB} \\ \hline \text{1 RR} & 0.6 & 0 & 0.4 & 0 \\ \text{2 RB} & 0.2 & 0 & 0.8 & 0 \\ \text{3 BR} & 0 & 0.6 & 0 & 0.4 \\ \text{4 BB} & 0 & 0.2 & 0 & 0.8 \end{array}$$

$$\begin{tikzcd} 1 \arrow[loop above] \arrow[d] & 2 \arrow[l] \arrow[dl, shift left] \\ 3 \arrow[ur, shift left] \arrow[r] & 4 \arrow[u] \arrow[loop right] \end{tikzcd}$$

Irreducible: yes, as $1 \to 3 \to 4 \to 2 \to 1$

Aperiodic: yes, as $1 \to 1$

(b) Write an expression for the probability of the following event: both Alice and Bob toss the red coin on day 9, and they both toss Heads. Do not evaluate.

$$P^9_{11} \cdot (0.6)^2$$

11-entry of P^9

(c) Bob pays Alice \$1 every time she tosses Heads but he tosses Tails (but no money changes hands otherwise). Write an expression for the probability that a payment occurs on Day 9. Do not evaluate.

$$P^9_{11} \cdot 0.6 \cdot 0.4 + P^9_{12} \cdot 0.6 \cdot 0.8 + P^9_{13} \, 0.2 \cdot 0.4 + P^9_{14} \cdot 0.2 \cdot 0.8$$

$$= P^9_{11} \cdot 0.24 + P^9_{12} \, 0.48 + P^9_{13} \cdot 0.08 + P^9_{14} \cdot 0.16$$

(d) Explain, with justification, how you would determine the long-term proportion of days on which a payment occurs. Do not carry out the computation.

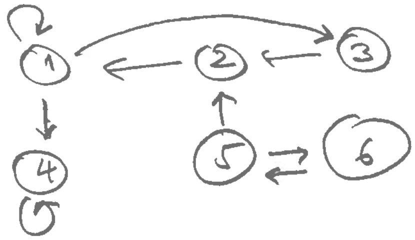
The chain has a unique invariant distribution $\pi = [\pi_1 \; \pi_2 \; \pi_3 \; \pi_4]$ that satisfies $\pi P = \pi$.

Answer:

$$\pi_1 \cdot 0.24 + \pi_2 \cdot 0.48 + \pi_3 \cdot 0.08 + \pi_4 \, 0.16$$

2. Determine the recurrent and transient classes of the Markov chain on six states with the following transition matrix:

$$
P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\!\!\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1/2 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}.
$$



Classes:

$\{1,2,3\}$ transient (as not closed)

$\{4\}$ recurrent (absorbing)

$\{5,6\}$ transient (as not closed)

3. Consider the branching process with the offspring distribution $p_0 = 1/6$, $p_2 = 1/2$, $p_3 = 1/3$. Let X_n be the population at generation n, where, as usual, the process starts with $X_0 = 1$, that is, with a single individual at generation 0.

(a) Find EX_4.

$$EX_1 = 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} = \underline{\underline{2}}$$

$$EX_4 = 2^4 = \underline{\underline{16}}$$

(b) Find the probability that the process dies out. (Observe: $2s^3 + 3s^2 - 6s + 1 = (s-1)(2s^2 + 5s - 1)$.)

$$\varphi(s) = \frac{1}{6} + \frac{1}{2}s^2 + \frac{1}{3}s^3$$

Solve $\varphi(s) = s$: $\quad \frac{1}{6} + \frac{1}{2}s^2 + \frac{1}{3}s^3 = s$

$$2s^3 + 3s^2 - 6s + 1 = 0$$

$$(s-1)(2s^2 + 5s - 1) = 0$$

$$2s^2 + 5s - 1 = 0 \qquad s = \frac{-5 \cancel{\pm} \sqrt{25+8}}{4}$$

$$= \underline{\underline{\frac{-5 + \sqrt{33}}{4} = \pi_0}}$$

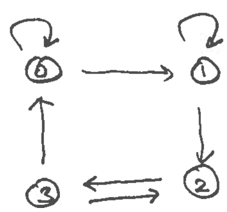
(b) Find the conditional probability that the process ever dies out, given that $X_4 = 7$.

$$\underline{\underline{\pi_0^7}}$$

4. The 4 vertices of a square are labeled, clockwise from top left, 0, 1, 2, 3. On day 0, Alice places a token on the vertex 0. On days 1, 2, 3, ... Alice decides that day's position of the token as follows. First, she tosses a coin with Heads probability $p \in (0,1)$. If the toss is Heads, Alice moves the token clockwise to the next vertex (for example, if the token is at 3, it is moved to 0). If the toss is Tails, then she proceeds as follows: if the token is at 0 or 1, it is not moved; if the token is at 3, it gets moved to 2; and if the token is at 3 it gets moved to 3.

(a) Write down the transition matrix P for this Markov chain. Is this chain irreducible? Aperiodic?

$$P = p \cdot \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ 1 & & & 0 \end{bmatrix} + (1-p)\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix} = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1-p & p & 0 & 0 \\ 0 & 1-p & p & 0 \\ 0 & 0 & 0 & 1 \\ p & 0 & 0 & 1-p \end{bmatrix} \end{array}$$



Irreducible : yes $(0 \to 1 \to 2 \to 3 \to 0)$

Aperiodic : yes $(0 \to 0)$

(b) For all states i and j, determine $\lim_{n\to\infty} P^n_{ij}$. Justify your answer. (No long calculations are necessary!)

The chain with finite space is irreducible and aperiodic, so it has a unique stationary distribution $\pi = [\pi_0 \; \pi_1 \; \pi_2 \; \pi_3]$, and $\lim_{n\to\infty} P^n_{ij} = \pi_j$, $\forall i, j$.
The matrix is doubly stochastic, so $\pi_j = \frac{1}{4}$, for all j.

Answer: $\lim_{n\to\infty} P^n_{ij} = 1/4$, $\forall i, j$.

(c) Each day, Alice wins i dollars if the token is at i. What are her long-term average daily winnings?

Answer: $\frac{1}{4}(0+1+2+3) = \frac{3}{2}$