Chapter 4

Topological Spaces

In the previous chapters, we discussed the convergence of sequences, the continuity of functions, and the compactness of sets. We expressed these properties in terms of a metric or norm. Some types of convergence, such as the pointwise convergence of real-valued functions defined on an interval, cannot be expressed in terms of a metric on a function space. Topological spaces provide a general framework for the study of convergence, continuity, and compactness. The fundamental structure on a topological space is not a distance function, but a collection of open sets; thinking directly in terms of open sets often leads to greater clarity as well as greater generality.

4.1 Topological spaces

Definition 4.1 A *topology* on a nonempty set X is a collection of subsets of X, called *open sets*, such that:

(a) the empty set $\emptyset$ and the set X are open;
(b) the union of an arbitrary collection of open sets is open;
(c) the intersection of a finite number of open sets is open.

A subset A of X is a *closed set* if and only if its complement, $A^c = X \setminus A$, is open.

More formally, a collection $\mathcal{T}$ of subsets of X is a topology on X if:

(a) $\emptyset, X \in \mathcal{T}$;
(b) if $G_\alpha \in \mathcal{T}$ for $\alpha \in \mathcal{A}$, then $\bigcup_{\alpha\in\mathcal{A}} G_\alpha \in \mathcal{T}$;
(c) if $G_i \in \mathcal{T}$ for $i = 1, 2 \ldots, n$, then $\bigcap_{i=1}^{n} G_i \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a *topological space*; if $\mathcal{T}$ is clear from the context, then we often refer to X as a topological space.

Example 4.2 Let X be a nonempty set. The collection $\{\emptyset, X\}$, consisting of the empty set and the whole set, is a topology on X, called the *trivial topology* or

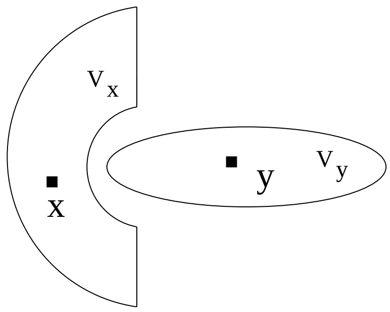

Fig. 4.1 The Hausdorff property.

indiscrete topology. The *power set* $\mathcal{P}(X)$ of X, consisting of all subsets of X, is a topology on X, called the *discrete topology*.

Example 4.3 Let (X, d) be a metric space. Then the set of all open sets defined in Definition 1.36 is a topology on X, called the *metric topology.* For instance, a subset G of $\mathbb{R}$ is open with respect to the standard, metric topology on $\mathbb{R}$ if and only if for every $x \in G$ there is an open interval I such that $x \in I$ and $I \subset G$.

Example 4.4 Let $(X, \mathcal{T})$ be a topological space and Y a subset of X. Then

$$\mathcal{S} = \{H \subset Y \mid H = G \cap Y \text{ for some } G \in \mathcal{T}\}$$

is a topology on Y. The open sets in Y are the intersections of open sets in X with Y. This topology is called the *induced* or *relative topology* of Y in X, and $(Y, \mathcal{S})$ is called a topological subspace of $(X, \mathcal{T})$. For instance, the interval $[0, 1/2)$ is an open subset of $[0, 1]$ with respect to the induced metric topology of $[0, 1]$ in $\mathbb{R}$, since $[0, 1/2) = (-1/2, 1/2) \cap [0, 1]$.

A set $V \subset X$ is a *neighborhood* of a point $x \in X$ if there exists an open set $G \subset V$ with $x \in G$. We do not require that V itself is open. A topology $\mathcal{T}$ on X is called *Hausdorff* if every pair of distinct points $x, y \in X$ has a pair of nonintersecting neighborhoods, meaning that there are neighborhoods V_x of x and V_y of y such that $V_x \cap V_y = \emptyset$ (see Figure 4.1). When the topology is clear, we often refer to X as a Hausdorff space. Almost all the topological spaces encountered in analysis are Hausdorff. For example, all metric topologies are Hausdorff. On the other hand, if X has at least two elements, then the trivial topology on X is not Hausdorff.

We can express the notions of convergence, continuity, and compactness in terms of open sets. Let X and Y be a topological spaces.

Definition 4.5 A sequence (x_n) in X *converges* to a limit $x \in X$ if for every neighborhood V of x, there is a number N such that $x_n \in V$ for all $n \geq N$.

This definition says that the sequence eventually lies entirely in every neighborhood of x.

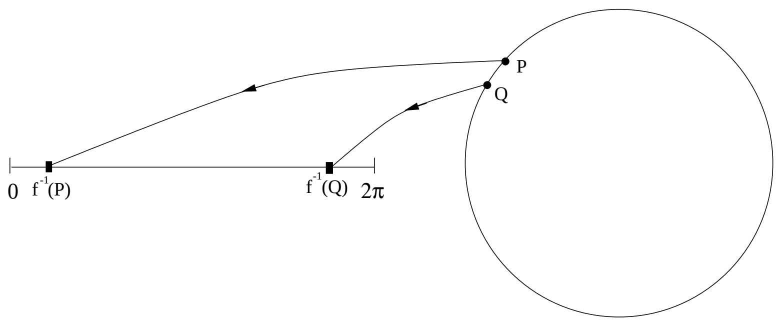

Fig. 4.2 The interval and the circle are not homeomorphic. There are arbitrarily close points on the circle, P and Q in the figure, which have inverse images near the left and right end points of the interval respectively. Hence, the inverse images are not close and the inverse map cannot be continuous.

Definition 4.6 A function $f : X \to Y$ is *continuous* at $x \in X$ if for each neighborhood W of $f(x)$ there exists a neighborhood V of x such that $f(V) \subset W$. We say that f is continuous on X if it is continuous at every $x \in X$.

Theorem 4.7 Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces and $f : X \to Y$. Then f is continuous on X if and only if $f^{-1}(G) \in \mathcal{T}$ for every $G \in \mathcal{S}$.

Thus, a continuous function is characterized by the property that the inverse image of an open set is open. We leave the proof to Exercise 4.4.

Definition 4.8 A function $f : X \to Y$ between topological spaces X and Y is a *homeomorphism* if it is a one-to-one, onto map and both f and f^{-1} are continuous. Two topological spaces X and Y are *homeomorphic* if there is a homeomorphism $f : X \to Y$.

Homeomorphic spaces are indistinguishable as topological spaces. For example, if $f : X \to Y$ is a homeomorphism, then G is open in X if and only if $f(G)$ is open in Y, and a sequence (x_n) converges to x in X if and only if the sequence $(f(x_n))$ converges to $f(x)$ in Y.

A one-to-one, onto map f always has an inverse f^{-1}, but f^{-1} need not be continuous even if f is.

Example 4.9 We define $f : [0, 2\pi) \to \mathbb{T}$ by $f(\theta) = e^{i\theta}$, where $[0, 2\pi) \subset \mathbb{R}$ with the topology induced by the usual topology on $\mathbb{R}$, and $\mathbb{T} \subset \mathbb{C}$ is the unit circle with the topology induced by the usual topology on $\mathbb{C}$. Then, as illustrated in Figure 4.2, f is continuous but f^{-1} is not.

Definition 4.10 A subset K of a topological space X is *compact* if every open cover of K contains a finite subcover.

It follows from the definition that a subset K of X is compact in the topology on X if and only if K is compact as a subset of itself with respect to the relative topology of K in X. This contrasts with the fact that a set $G \subset Y$ may be relatively open in Y, yet not be open in X. For this reason, while we define the notion of relatively open, we do not define the notion of relatively compact.

4.2 Bases of open sets

The collection of all open sets in a topological space is often huge and unwieldy. The topological properties of metric spaces can be expressed entirely in terms of open balls, which form a rather small subset of the open sets. In this section we introduce subsets of a topological space that play a similar role to open balls in a metric space.

Definition 4.11 A subset $\mathcal{B}$ of a topology $\mathcal{T}$ is a *base* for $\mathcal{T}$ if for every $G \in \mathcal{T}$ there is a collection of sets $B_\alpha \in \mathcal{B}$ such that $G = \bigcup_\alpha B_\alpha$. A collection $\mathcal{N}$ of neighborhoods of a point $x \in X$ is called a *neighborhood base* for x if for each neighborhood V of x there is a neighborhood $W \in \mathcal{N}$ such that $W \subset V$. A topological space X is *first countable* if every $x \in X$ has a countable neighborhood base, and *second countable* if X has a countable base.

Example 4.12 The collection of all open intervals (a, b) with $a, b \in \mathbb{R}$ is a base for the standard topology on $\mathbb{R}$. The collection of all open intervals $(a, b) \subset \mathbb{R}$ with rational endpoints $a, b \in \mathbb{Q}$ is a countable base for the standard topology on $\mathbb{R}$. Thus, the standard topology is second countable.

Example 4.13 Let X be a metric space and A a dense subspace of X. The set of open balls $B_{1/n}(x)$, with $n \geq 1$ and $x \in A$ is a base for the metric topology on X. A metric space is first countable, and a separable metric space is second countable.

Example 4.14 If X is topological space with the discrete topology, then the collection of open sets

$$\mathcal{B} = \{\{x\} \mid x \in X\}$$

is a base. The discrete topology is first countable, and if X is countable, then it is second countable.

It is often useful to define a topology in terms of a base.

Theorem 4.15 A collection of open sets $\mathcal{B} \subset \mathcal{T}$ is a base for the topology $\mathcal{T}$ on a set X if and only if $\mathcal{B}$ contains a neighborhood base for x for every $x \in X$.

Proof. Suppose $\mathcal{B}$ is a base for $\mathcal{T}$. If N is a neighborhood of $x \in X$, then there is an open set $G \in \mathcal{T}$ such that $x \in G \subset N$. Since $\mathcal{B}$ is a base, there are sets $B_\alpha \in \mathcal{B}$

such that $\bigcup_\alpha B_\alpha = G$. Therefore, there is an α such that $x \in B_\alpha$ and $B_\alpha \subset N$. It follows that $\mathcal{B}$ contains a neighborhood base for x.

Conversely, if a collection of open sets $\mathcal{B}$ contains a neighborhood base for every $x \in X$, then for every open set $G \in \mathcal{T}$ and every $x \in G$ there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset G$. Therefore, $\bigcup_x B_x = G$, so $\mathcal{B}$ is a base for the topology. $\square$

Example 4.16 Suppose that X is the space of all real-valued functions on the interval $[a, b]$. We may identify a function $f : [a, b] \to \mathbb{R}$ with a point $\prod_{x \in [a,b]} f(x)$ in $\mathbb{R}^{[a,b]}$, so $X = \mathbb{R}^{[a,b]}$ is the $[a, b]$-fold Cartesian product of $\mathbb{R}$. Let $x = \{x_1, \ldots, x_n\}$, where $x_i \in [a, b]$, and $y = \{y_1, \ldots, y_n\}$, where $y_i \in \mathbb{R}$, be finite subsets of $[a, b]$ and $\mathbb{R}$, respectively. For $\epsilon > 0$, we define a subset $B_{x,y,\epsilon}$ of X by

$$B_{x,y,\epsilon} = \{f \in X \mid |f(x_i) - y_i| < \epsilon \text{ for } i = 1, \ldots, n\}. \tag{4.1}$$

The topology of pointwise convergence is the smallest topology on X that contains the sets $B_{x,y,\epsilon}$ for all finite sets $x \subset [a, b]$, $y \subset \mathbb{R}$, and $\epsilon > 0$. We have $f_n \to f$ with respect to this topology if and only if $f_n(x) \to f(x)$ for every $x \in [a, b]$. If $f \in X$ and $y_i = f(x_i)$, then the sets $B_{x,y,\epsilon}$ form a neighborhood base for $f \in X$. This topology is not first countable.

The set $B_{x,y,\epsilon}$ in (4.1) is called a *cylinder set*. It has a rectangular base

$$(y_1 - \epsilon, y_1 + \epsilon) \times (y_2 - \epsilon, y_2 + \epsilon) \times \ldots \times (y_n - \epsilon, y_n + \epsilon)$$

in the $x_1, x_2, \ldots, x_n$ coordinates, and is unrestricted in the other coordinate directions. More picturesquely, $B_{x,y,\epsilon}$ is sometimes called a "slalom set," because it consists of all functions whose graphs pass through the "slalom gates" at x_i with radius ϵ and center y_i.

A base for the topology of pointwise convergence is given by all finite intersections of sets of the form $B_{x,y,\epsilon}$. In fact, it is sufficient to take the sets of the form

$$\{f \in X \mid |f(x_i) - y_i| < \epsilon_i \text{ for } i = 1, \ldots, n\} \tag{4.2}$$

where $n \in \mathbb{N}$, $\{x_1, \ldots, x_n\} \subset [a, b]$, $\{y_1, \ldots, y_n\} \subset \mathbb{R}$, and $\epsilon_i > 0$. The sets of functions in (4.2) with intervals of variable width $\epsilon_i > 0$ generate the same topology as the sets with intervals of a fixed width because $B_{x,y,\epsilon}$ with $\epsilon = \min \epsilon_i > 0$ is contained inside the set in (4.2).

We say that a topological space $(X, \mathcal{T})$ is *metrizable* if there is a metric on X whose metric topology is $\mathcal{T}$. For a metrizable space, we can give sequential characterizations of compact sets (Theorem 1.62), closed sets (Proposition 1.41), and continuous functions (Proposition 1.34). These sequential characterizations may not apply in a nonmetrizable topological space. There is, however, a generalization of sequences, called *nets*, that can be used to express all the above properties in an analogous way [12]. We will not make use of nets in this book.

For example, the closure $\overline{A}$ of a subset A of a topological space X is the smallest closed set that contains A. If X is metrizable, then $\overline{A}$ is the set of limits of convergent sequences whose terms are in A (see Section 1.5), but if X is a not metrizable, then this procedure may fail. We call the set of limit points of sequences in A the *sequential closure* of A and denote it by $\overline{A}^S$. The sequential closure is a subset of the closure, but it may be a strict subset, as illustrated by the following example.

Example 4.17 Consider the space of all functions $f : [0,1] \to \mathbb{R}$ with the topology of pointwise convergence. For each $m, n \geq 1$, we let

$$f_{m,n}(x) = [\cos(m!\pi x)]^{2n}.$$

We define functions f_m and f by the pointwise limits,

$$f_m(x) = \lim_{n\to\infty} f_{m,n}(x) = \begin{cases} 1 & \text{if } x = k/m!,\ k = 0, \dots, m!, \\ 0 & \text{otherwise,} \end{cases}$$

$$f(x) = \lim_{m\to\infty} f_m(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{f_{m,n} \mid m, n \geq 1\}$. Then these limits show that

$$f_m \in \overline{A}^S, \qquad f \in \overline{\overline{A}^S}^S.$$

It is possible to show that the pointwise limit of a sequence of continuous functions on $[0,1]$ is continuous on a dense subset of $[0,1]$. Since f is nowhere continuous in $[0,1]$, it is not the pointwise limit of any subsequence of the continuous functions $f_{m,n}$. Therefore, $f \in \overline{A}$ but $f \notin \overline{A}^S$. This example shows that the topology of pointwise convergence on the real-valued functions on $[0,1]$ is not metrizable.

A linear space with a topology defined on it, which need not be derived from a norm or metric, such that the operations of vector addition and scalar multiplication are continuous is called a *topological linear space*, or a *topological vector space*. The space of real-valued functions on a set with the topology of pointwise convergence is an example of a topological linear space. Topological linear spaces, such as the Schwartz space, also arise in connection with distribution theory (see Chapter 11).

4.3 Comparing topologies

Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two topologies on the same space X. Then $\mathcal{T}_2$ is said to be *finer* or *stronger* than $\mathcal{T}_1$ if $\mathcal{T}_1 \subset \mathcal{T}_2$, meaning that $\mathcal{T}_2$ has more open sets; we also say that $\mathcal{T}_1$ is *coarser* or *weaker* than $\mathcal{T}_2$. If $\mathcal{T}_1$ is stronger than $\mathcal{T}_2$, then $x_n \to x$ with respect to $\mathcal{T}_1$ implies that $x_n \to x$ with respect to $\mathcal{T}_2$. For example, the strongest topology on any set is the discrete topology, and a sequence converges with respect to the discrete topology if and only if it is eventually constant. The weakest topology on any set is the trivial topology, and every sequence converges with respect to the trivial topology. It is possible that two topologies $\mathcal{T}_1$, $\mathcal{T}_2$ are not comparable, meaning that $\mathcal{T}_1$ is neither finer nor coarser than $\mathcal{T}_2$.

Proposition 4.18 Let X and Y be two spaces, each with two topologies, $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{S}_1, \mathcal{S}_2$ respectively. Suppose that $f : (X, \mathcal{T}_1) \to (Y, \mathcal{S}_1)$ is a map from X to Y that is continuous with respect to the indicated topologies.

(a) If $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, then $f : (X, \mathcal{T}_2) \to (Y, \mathcal{S}_1)$ is continuous.
(b) If $\mathcal{S}_2$ is coarser than $\mathcal{S}_1$, then $f : (X, \mathcal{T}_1) \to (Y, \mathcal{S}_2)$ is continuous.

Proof. These statements are a direct consequence of the general definition of continuity in Definition 4.6. □

The identity map $I : (X, \mathcal{T}) \to (X, \mathcal{T})$, where $I(x) = x$, is a homeomorphism when we use the same topology $\mathcal{T}$ on the domain and range. This is not true when we use two different topologies on X. For example, the identity map from a set X containing at least two elements equipped with the trivial topology to the set X equipped with the discrete topology,

$$I : (X, \{\emptyset, X\}) \to (X, \mathcal{P}(X)),$$

is discontinuous at every point $x \in X$. As the following theorems show, the identity map is a useful tool for comparing topologies on a set.

Theorem 4.19 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then the identity map $I : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

Proof. This is a direct consequence of Theorem 4.7. □

Corollary 4.20 The identity map $I : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a homeomorphism if and only if $\mathcal{T}_1 = \mathcal{T}_2$.

Theorem 4.21 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then the equality of $\mathcal{T}_1$ and $\mathcal{T}_2$ is equivalent to the following condition: for all topological spaces $(Y, \mathcal{S})$, a function $f : (X, \mathcal{T}_1) \to (Y, \mathcal{S})$ is continuous if and only if the function $f : (X, \mathcal{T}_2) \to (Y, \mathcal{S})$ is continuous.

Proof. If $\mathcal{T}_1 = \mathcal{T}_2$, then the condition about continuous functions $f : X \to Y$ is trivial. Conversely, taking $(Y, \mathcal{S}) = (X, \mathcal{T}_1)$ and $(Y, \mathcal{S}) = (X, \mathcal{T}_2)$, we see that the condition implies that $I : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a homeomorphism, so $\mathcal{T}_1 = \mathcal{T}_2$ from Corollary 4.20. □

A topology is often defined by the specification of a neighborhood base at each point. We therefore want to compare topologies in terms of their neighborhood bases.

Theorem 4.22 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Suppose that for each $x \in X$ there are neighborhood bases $\mathcal{N}_1$ and $\mathcal{N}_2$ of x for $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, such that for every $V_1 \in \mathcal{N}_1$ there is a $V_2 \in \mathcal{N}_2$ with $V_2 \subset V_1$. Then $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.

Proof. The hypothesis of the theorem implies that $I : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous, so the result follows from Theorem 4.19. □

Corollary 4.23 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then $\mathcal{T}_1 = \mathcal{T}_2$ if and only if for each $x \in X$ there are neighborhood bases $\mathcal{M}_1$ and $\mathcal{M}_2$ of x for $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, such that for every $V_1 \in \mathcal{M}_1$ there is a $V_2 \in \mathcal{M}_2$ with $V_2 \subset V_1$, and there are neighborhood bases $\mathcal{N}_1$ and $\mathcal{N}_2$ of x for $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, such that for every $W_2 \in \mathcal{N}_2$ there is a $W_1 \in \mathcal{N}_1$ with $W_1 \subset W_2$.

Different metrics, or norms, on a space X can lead to the same topology. For example, this is certainly the case if d_1 and d_2 are two metrics on X such that $d_1(x, y) = 2d_2(x, y)$ for all $x, y \in X$. More generally, if two metrics lead to the same set of convergent sequences, then all their topological properties are the same.

Theorem 4.24 Two metric topologies, defined by two metrics on the same space, are equal if and only if they have the same collection of convergent sequences with the same limits.

Proof. The proof is a direct application of Corollary 4.20 and the sequential characterization of continuity on metric spaces. □

4.4 References

In this chapter, we have limited our discussion to the basic definitions of point set topology. For more information, see Kelley [28] and Rudin [48].

4.5 Exercises

Exercise 4.1 Suppose that K is a compact subspace of a Hausdorff space. Prove that K is closed. Show that this result need not be true if X is not Hausdorff.

Exercise 4.2 If A is a subset of a topological space, then the *interior* A° of A is the union of all open sets contained in A, the *closure* $\overline{A}$ of A is the intersection of all closed sets that contain A, and the *boundary* ∂A of A is defined by $\partial A = \overline{A} \bigcap \overline{A^c}$. Show that a set is closed if and only if it contains its boundary, and open if and only if it is disjoint from its boundary. What are the closure, interior, and boundary of the Cantor set in $\mathbb{R}$ with its usual topology?

Exercise 4.3 Let (X, d_1) and (Y, d_2) be metric spaces. Prove that the topological definitions of convergence and continuity are equivalent to the metric space definitions in Definitions 1.12 and 1.26.

Exercise 4.4 Prove Theorem 4.7.

Exercise 4.5 A topological space is *connected* if it is not the disjoint union of two non-empty open sets.

(a) What are the connected subsets of $\mathbb{R}$?
(b) Show that $X \times Y$ is connected if X and Y are connected.

Exercise 4.6 Show that $\mathbb{R}$ is homeomorphic to $(0, 1)$, but not to $\mathbb{R}^2$.
HINT. Show that $\mathbb{R}^2$ remains connected when one point is removed.