ADVANCED CALCULUS

Math 127B, Winter 2005

Midterm 1

NAME..

I.D. NUMBER...

No books, notes, or calculators. Show all your work.
Give complete proofs of all your answers.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (a) [15%] Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2} x^n.$$

(b) [5%] Determine all points $x \in \mathbb{R}$ where the series converges.

2. [20%] Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} = 1 - \frac{1}{2 \cdot 1} x^2 + \frac{1}{2^2 \cdot 2!} x^4 - \frac{1}{2^3 \cdot 3!} x^6 + \cdots.$$

(You can assume that this power series converges for all $x \in \mathbb{R}$.) Prove that $f(x)$ satisfies the following initial value problem for an ordinary differential equation:

$$\begin{aligned} f' + xf &= 0, \\ f(0) &= 1. \end{aligned}$$

3. (a) [10%] Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{x}{1 + nx}.$$

What is the pointwise limit of the sequence (f_n) as $n \to \infty$?

(b) [10%] Does (f_n) converge uniformly on $[0,1]$? Justify your answer.

4. (a) [15%] Prove that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{n^2 + x^2}$$

converges uniformly on $[0, 1]$.

(b) [5%] Prove that

$$\int_0^1 f(x)\,dx = \frac{1}{2} \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n^2}\right).$$

5. (a) [15%] Suppose that (f_n) is a sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$ that converges uniformly as $n \to \infty$ to a function $f : [a, b] \to \mathbb{R}$. If (x_n) is a sequence of points in $[a, b]$ such that $x_n \to a$ as $n \to \infty$, prove that $\lim_{n\to\infty} f_n(x_n) = f(a)$. HINT: f is continuous at a.

(b) [5%] Give an example to show that this result need not be true if (f_n) converges to f pointwise.