Advanced Calculus
Math 127B, Winter 2005
Solutions: Midterm 1

1. (a) [15%] Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2} x^n.$$

(b) [5%] Determine all points $x \in \mathbb{R}$ where the series converges.

Solution.

- (a) By the root test, he radius of convergence R is

$$\begin{aligned} R &= \frac{1}{\limsup_{n\to\infty} |3^n/n^2|^{1/n}} \\ &= \frac{1}{3 \limsup_{n\to\infty} (n^{1/n})^{-2}} \\ &= \frac{1}{3}, \end{aligned}$$

  since $n^{1/n} \to 1$ as $n \to \infty$. (The ratio test can be used instead.)

- (b) At $x = 1/3$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

  and at $x = -1/3$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

  Both of these series converge, so the original power series converges for $-1/3 \leq x \leq 1/3$ and diverges for $|x| > 1/3$.

2. [20%] Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} x^{2n} = 1 - \frac{1}{2 \cdot 1} x^2 + \frac{1}{2^4 \cdot 2!} x^4 - \frac{1}{2^6 \cdot 3!} x^6 + \ldots.$$

(You can assume that this power series converges for all $x \in \mathbb{R}$.) Prove that $f(x)$ satisfies the following initial value problem for an ordinary differential equation:

$$\begin{aligned} f' + xf &= 0, \\ f(0) &= 1. \end{aligned}$$

Solution.

- A power series is differentiable inside its interval of convergence and the derivative is given by term-by-term differentiation. Therefore, the function f is differentiable on $\mathbb{R}$, and

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} 2n x^{2n-1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n-1}(n-1)!} x^{2n-1} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{(m+1)}}{2^m m!} x^{2m+1} \\ &= -x \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} x^{2m} \\ &= -xf(x), \end{aligned}$$

  where we have made the change of summation variable $n = m+1$. This proves that f satisfies the differential equation $f' + xf = 0$. Setting $x = 0$ in the power series for f, we get that $f(0) = 1$.

3. (a) [10%] Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{x}{1+nx}.$$

What is the pointwise limit of the sequence (f_n) as $n \to \infty$?
(b) [10%] Does (f_n) converge uniformly on $[0,1]$? Justify your answer.

Solution.

- (a) We have
$$f_n(x) = \frac{1}{n}\frac{x}{x+1/n}.$$
If $x \neq 0$, then $x/(x+1/n) \to 1$ as $n \to \infty$, and $f_n(x) \to 0$ since $1/n \to 0$. If $x = 0$ then $f_n(0) = 0 \to 0$ as $n \to \infty$. Thus (f_n) converges pointwise to 0 as $n \to \infty$.

- (b) This convergence is uniform. Given $\epsilon > 0$, let $N = 1/\epsilon$. Then, since
$$\left|\frac{x}{x+1/n}\right| \leq 1 \quad \text{for every } x \in [0,1],$$
we have for $n > N$ that
$$|f_n(x)| \leq \frac{1}{n}\left|\frac{x}{x+1/n}\right| \leq \frac{1}{n} < \epsilon \quad \text{for every } x \in [0,1].$$

4. (a) [15%] Prove that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{n^2 + x^2}$$

converges uniformly on $[0, 1]$.
(b) [5%] Prove that

$$\int_0^1 f(x)\, dx = \frac{1}{2} \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n^2}\right).$$

Solution.

- (a) For $x \in [0, 1]$, we have

  $$\left|\frac{x}{n^2 + x^2}\right| \leq \frac{1}{n^2 + x^2} \leq \frac{1}{n^2}.$$

  Since

  $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

  converges, the Weierstrass M-test implies that the series converges uniformly.

- (b) Since the series converges uniformly, the sum is continuous, and we can evaluate the integral by integrating the series term-by-term. Hence

$$\begin{aligned}
\int_0^1 f(x)\, dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{x}{n^2 + x^2}\, dx \\
&= \sum_{n=1}^{\infty} \left[\frac{1}{2} \log\left(n^2 + x^2\right)\right]_0^1 \\
&= \sum_{n=1}^{\infty} \frac{1}{2} \left[\log\left(n^2 + 1\right) - \log\left(n^2\right)\right] \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n^2}\right).
\end{aligned}$$

5. (a) [15%] Suppose that (f_n) is a sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$ that converges uniformly as $n \to \infty$ to a function $f : [a, b] \to \mathbb{R}$. If (x_n) is a sequence of points in $[a, b]$ such that $x_n \to a$ as $n \to \infty$, prove that

$$\lim_{n\to\infty} f_n(x_n) = f(a).$$

(b) [5%] Give an example to show that this result need not be true if (f_n) converges to f pointwise.

Solution.

- (a) Let $\epsilon > 0$ be given. Since (f_n) converges uniformly to f, there exists N_1 such that $n > N_1$ implies that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \text{for all } x \in [a, b].$$

Since the f_n are continuous and converge uniformly, the function f is continuous. Hence there exists a $\delta > 0$ such that $|x - a| < \delta$ implies that

$$|f(x) - f(a)| < \frac{\epsilon}{2}$$

Since $x_n \to a$ as $n \to \infty$, there exists N_2 such that $|x_n - a| < \delta$ for all $n > N_2$. Let $N = \max\{N_1, N_2\}$. Then if $n > N$, it follows that

$$|f_n(x_n) - f(a)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which implies that $f_n(x_n) \to f(a)$ as $n \to \infty$.

- (b) Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \frac{nx}{1 + nx}.$$

Then (f_n) converges pointwise to the function f given by

$$f(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Let $x_n = 1/n$. Then $x_n \to 0$ as $n \to \infty$ and $f_n(x_n) = 1/2 \to 1/2$ as $n \to \infty$, but $f(0) = 1$.