Advanced Calculus

Math 127B, Winter 2005

Midterm 2

NAME..

I.D. NUMBER...

No books, notes, or calculators. Show all your work.
Except in Question 1, give complete proofs of all your answers.
You can use any standard theorem, provided you state it carefully.

Question	Points	Score
1	10	
2	15	
3	15	
4	20	
5	20	
6	20	
Total	100	

1. [10%] For each of the following statements, say if it is true or false. (No explanation is required.)

(a) If f is differentiable at an interior point x of its domain and $f'(x) = 0$, then f has a local maximum or minimum at x.

(b) If f has a local maximum or minimum at an interior point x of its domain and f is differentiable at x, then $f'(x) = 0$.

(c) If f is differentiable in an open interval, then f is continuous in the interval.

(d) If f is differentiable in an open interval, then f' is continuous in the interval.

2. [15%] State the mean value theorem. Prove that

$$|\sin x - \sin y| \leq |x - y| \qquad \text{for all } x, y \in \mathbb{R}.$$

3. [15%] Use L'Hospital's rule to evaluate the following limit:

$$\lim_{x\to 0^+} \frac{\log(-\log x)}{\log x}.$$

Justify your steps.

4. [20%] Let

$$f(x) = \begin{cases} x\sin(1/x^3) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

$$g(x) = \begin{cases} x^3\sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

(a) Is $f(x)$ differentiable at $x = 0$? If so, what is $f'(0)$?

(b) Is $g(x)$ differentiable at $x = 0$? If so, what is $g'(0)$?

5. [20%] (a) Consider the sine function $\sin x$ defined on the open interval $-\pi/2 < x < \pi/2$. Show that this function is strictly increasing and hence invertible. What is the domain of the inverse function?

(b) Prove that the inverse function is differentiable, and compute its derivative.

6. [20%] (a) Let $f(x) = \log(1+x)$ for $x > -1$. Use a proof by induction to show that for $n \geq 1$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}.$$

(b) Write out the Taylor series of f (at $x = 0$).

(c) Assume that $x > 0$. Give an expression for the remainder $R_n(x)$ between $f(x)$ and its Taylor polynomial of degree $n-1$ involving an intermediate point $0 < y < x$.

(d) Prove that the Taylor series converges to $f(x)$ if $0 < x \leq 1$.