**Sample Midterm Solutions**
**Math 127B. Winter, 2005**

**1.** Consider the sequence $(f_n)$ of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{nx}{\sqrt{1 + n^2x^2}}.$$

Find the pointwise limit of this sequence as $n \to \infty$. Does the sequence converge uniformly on $\mathbb{R}$? Justify your answer.

**Solution.**

- For $x > 0$,

$$f_n(x) = \frac{1}{\sqrt{1/(n^2x^2) + 1}} \to 1 \quad \text{as } n \to \infty.$$

For $x < 0$,

$$f_n(x) = \frac{-1}{\sqrt{1/(n^2x^2) + 1}} \to -1 \quad \text{as } n \to \infty.$$

For $x = 0$,

$$f_n(0) = 0 \to 0 \quad \text{as } n \to \infty.$$

Hence $f_n \to f$ pointwise as $n \to \infty$, where

$$f(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

- The sequence cannot converge uniformly because the $f_n$ are continuous, $f$ is discontinuous, and the uniform limit of continuous functions is continuous.

**2.** Let

$$f_n(x) = \frac{nx + \sin(nx^2)}{n}.$$

Prove that the following limit exists, and compute its value:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx.$$

**Solution.**

- The sequence $(f_n)$ of continuous functions converges uniformly to the the function $x$ on $[0, 1]$. To prove this, suppose $\epsilon > 0$, and choose $N = 1/\epsilon$. Then if $n > N$, we have that

$$|f_n(x) - x| = \left|\frac{\sin(nx^2)}{n}\right| \le \frac{1}{n} < \epsilon \quad \text{for all } x \in [0, 1].$$

  If $(f_n)$ is a sequence of continuous functions and $f_n \to f$ uniformly on $[a, b]$, then

$$\int_a^b f_n(x)\, dx \to \int_a^b f(x)\, dx.$$

  Therefore for the sequence given in the problem

$$\int_0^1 f_n(x)\, dx \to \int_0^1 x\, dx = \frac{1}{2}.$$

**3.** Prove that the following series

$$f(x) = \sum_{n=1}^{\infty} \frac{n^2 + x^4}{n^4 + x^2}$$

converges to a continuous function $f : \mathbb{R} \to \mathbb{R}$.

**Solution.**

- Suppose $R > 0$. Then for all $x \in [-R, R]$ we have

$$\begin{aligned} \left| \frac{n^2 + x^4}{n^4 + x^2} \right| &\leq \frac{n^2 + x^4}{n^4} \\ &\leq \frac{1}{n^2} + \frac{R^4}{n^4}. \end{aligned}$$

  The series

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} + \frac{R^4}{n^4} \right\} = \sum_{n=1}^{\infty} \frac{1}{n^2} + R^4 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

  converges, so the Weierstrass M-test implies that the series for $f$ converges uniformly on the bounded interval $[-R, R]$. The terms in the series are continuous and the uniform limit of continuous functions is continuous, so $f$ is continuous on $[-R, R]$ for every $R > 0$. Since every $x \in \mathbb{R}$ lies in such an interval for sufficiently large $R$, it follows that $f$ is continuous on $\mathbb{R}$.

- Note that the series does not converge uniformly on $\mathbb{R}$, so we can't use the argument that the sum is continuous on $\mathbb{R}$ because the series converges uniformly on $\mathbb{R}$.

**4.** Determine the radius of convergence $R$ of the power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots . \end{aligned}$$

Where does the series converge? Prove that

$$f'(x) = \frac{1}{1+x^2} \qquad \text{in } |x| < R.$$

**Solution.**

- By the root test, and the standard limit $n^{1/n} \to 1$ as $n \to \infty$, the radius of convergence is

$$R = \limsup_{n\to\infty} \left( \frac{1}{2n+1} \right)^{1/(2n+1)} = 1.$$

(Alternatively, the ratio test gives the same result.) Hence the series converges in $|x| < 1$ and diverges in $|x| > 1$. At $x = 1$, the series is

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots,$$

which converges by the alternating series test. If $x = -1$, the series is

$$-\left( 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \right),$$

which diverges by comparison with the divergent harmonic series, since

$$\begin{aligned} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots &> \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots \\ &> \frac{1}{2}\left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \right) \end{aligned}$$

Thus, the series converges for $-1 < x \le 1$.

- A power series is differentiable inside its interval of convergence and can be differentiated term-by-term. Hence, in $|x| < 1$ we have

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \\ &= \sum_{n=0}^{\infty} (-x^2)^n \\ &= \frac{1}{1+x^2}, \end{aligned}$$

where we have used the standard sum of a geometric series,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad |a| < 1,$$

with $a = -x^2$.

**5.** Suppose that $(f_n)$ is a sequence of functions $f_n : [-1, 1] \to \mathbb{R}$ that converges uniformly on $[-1, 1]$ to a function $f : \mathbb{R} \to \mathbb{R}$. If the limit

$$\lim_{x\to 0} f_n(x) = a_n$$

exists for each $n \in \mathbb{N}$, and the limit

$$\lim_{n\to\infty} a_n = a$$

exists, prove that $\lim_{x\to 0} f(x)$ exists, and

$$\lim_{x\to 0} f(x) = a$$

Give a counter-example to show that this result need not be true if $(f_n)$ converges to $f$ pointwise, but not uniformly.

**Solution.**

- Let $\epsilon > 0$ be given. From the uniform convergence, we can choose $N_1$ such that $n > N_1$ implies that

  $$|f_n(x) - f(x)| < \frac{\epsilon}{3} \quad \text{for all } x \in [-1, 1].$$

  From the existence of the limit of $(a_n)$, we can choose $N_2$ such that $n > N_2$ implies that

  $$|a_n - a| < \frac{\epsilon}{3}.$$

  Choose some $N > \max\{N_1, N_2\}$. The existence of the limit of $f_N(x)$ as $x \to 0$ implies that there exists $\delta > 0$ such that

  $$|f_N(x) - a_N| < \frac{\epsilon}{3} \quad \text{when } 0 < |x| < \delta.$$

  It follows that if $0 < |x| < \delta$, then

  $$\begin{aligned} |f(x) - a| &\leq |f(x) - f_N(x)| + |f_N(x) - a_N| + |a_N - a| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &< \epsilon, \end{aligned}$$

  which proves that $f(x) \to a$ as $x \to 0$.

- Consider the sequence of functions on $[-1, 1]$ defined by

$$f_n(x) = (1 - x^2)^n.$$

Then $f_n \to f$ pointwise as $n \to \infty$, where

$$f(x) = \begin{cases} 0 & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

By the continuity of $f_n$, we have

$$a_n = \lim_{x \to 0} f_n(x) = 1$$

for every $n$, and $a_n \to 1$ as $n \to \infty$. On the other hand,

$$\lim_{x \to 0} f(x) = 0 \neq 1.$$

- This result says that *uniform* convergence allows us to exchange the order of the limits:

$$\lim_{x \to 0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to 0} f_n(x)$$