Amalgamation of Heegaard splittings is unique

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Abstract

We show that amalgamation of Heegaard splittings is unique.

The notion of amalgamation of Heegaard splittings has been used implicitly for at least 15 years, but was first formalized by the author in [6]. Recently, the question as to whether or not amalgamation of Heegaard splittings is unique has received closer scrutiny. This note is an elaboration of [6, Proposition 2.8]. In particular, we establish the uniqueness of amalgamations of Heegaard splittings.

For standard definitions and results pertaining to 3-manifolds, see [2] or [3].

Definition 1. *A compression body is a* 3*-manifold* W *obtained from a closed orientable surface* S *by attaching* 2*-handles to* $S \times \{0\} \subset S \times I$ *and capping off any resulting* 2*-sphere boundary components with* 3*-handles. We denote* $S \times \{1\}$ *by* $\partial_+ W$ *and* $\partial W - \partial_+ W$ *by* $\partial_- W$*. Dually, a compression body is an orientable* 3*-manifold obtained from a closed orientable surface* $\partial_- W \times I$ *or a* $3-$*ball or a union of the two by attaching* 1*-handles.*

In the case where $\partial_- W = \emptyset$*, we also call* W *a handlebody.*

Definition 2. *A set of defining disks for a compression body* W *is a set of disks* $\{D_1, \ldots, D_n\}$ *properly embedded in* W *with* $\partial D_i \subset \partial_+ W$ *for* $i = 1, \ldots, n$ *such that the result of cutting* W *along* $D_1 \cup \cdots \cup D_n$ *is homeomorphic to* $\partial_- W \times I$ *or a* 3*-ball in the case that* W *is a handlebody.*

Definition 3. *A Heegaard splitting of a* 3*-manifold* M *is a pair* (V, W) *in which* V*,* W *are compression bodies and such that* $M = V \cup W$ *and* $V \cap W = \partial_+ V = \partial_+ W = S$*. We call* S *the splitting surface or Heegaard surface. Two Heegaard splittings are considered equivalent if their splitting surfaces are isotopic.*

The definition of amalgamation is a lengthy one. It was formally introduced by the author in [6], though it had been used implicitly by Casson, Gordon, Boileau, Otal and others. See for instance [1]. The general idea is as follows: A pair of 3-manifolds M_1, M_2 each with a Heegaard splitting are identified along components of their boundary. This results in a 3-manifold M. The Heegaard splittings of M_1, M_2 are used to construct a Heegaard splitting of M called the amalgamation of the two Heegaard splittings. One assumes that in each of M_1, M_2 the boundary components along which the gluing occurs are contained in a single compression body. Roughly speaking, the collars of the boundary components lying in this compression body are discarded and the remnants of the two compression bodies in $M_1 - collars$ identified to the remnants of the two compression bodies in $M_2 - collars$. This is done in such a way that the 1-handles that are attached to the collar on such a boundary component in M_1 become attached to the compression body in M_2 that does not meet any of the boundary components along which the gluing takes place and vice versa. For a formal definition see below.

Definition 4. *Let M_1, M_2 be 3-manifolds with R a closed subsurface of ∂M_1 and S a closed subsurface of ∂M_2. Suppose that R is homeomorphic to S via a homeomorphism h. Let $(X_1, Y_1), (X_2, Y_2)$ be Heegaard splittings of M_1, M_2. Choose $N(R) \subset X_1, N(S) \subset X_2$ such that this inclusion has the property that for some $R' \subset \partial M_1 \backslash R$ and $S' \subset \partial M_2 \backslash S$, $X_1 = N(R \cup R') \cup (1-handles)$ and $X_2 = N(S \cup S') \cup (1-handles)$. To keep track of our choices, we denote the particular choice of $N(R)$ by N_r and the particular choice of $N(S)$ by N_s. Here N_r is homeomorphic to $R \times I$ via a homeomorphism f and N_s is homeomorphic to $S \times I$ via a homeomorphism g. Let $\sim$ be the equivalence relation on $M_1 \cup M_2$ generated by*

(1) $x \sim y$ if $x, y \,\epsilon\, \eta(R)$ and $p_1 \cdot f(x) = p_1 \cdot f(y)$,

(2) $x \sim y$ if $x, y \,\epsilon\, \eta(S)$ and $p_1 \cdot g(x) = p_1 \cdot g(y)$,

(3) $x \sim y$ if $x \,\epsilon\, R$, $y \,\epsilon\, S$ and $h(x) = y$,

where p_1 is projection onto the first coordinate. Perform isotopies so that for D an attaching disk for a 1-handle in X_1, D' an attaching disk for a 1-handle in X_2, $[D] \cap [D'] = \emptyset$. Set $M = (M_1 \cup M_2)/\sim$, $X = (X_1 \cup Y_2)/\sim$, and $Y = (Y_1 \cup X_2)/\sim$. In particular, $(N_r \cup N_s/\sim) \cong R, S$. Then $X = Y_2 \cup N(R') \cup (1-handles)$, where the 1-handles are attached to $\partial_+ Y_2$ and connect $\partial N(R')$ to $\partial_+ Y_2$. Hence X is a compression body. Analogously, Y is a compression body. So (X, Y) is a Heegaard splitting of M. The splitting (X, Y) is called the amalgamation of (X_1, Y_1) and (X_2, Y_2) along R, S via h.

Proposition 1. *Let (X_1, Y_1) and (X_2, Y_2) be Heegaard splittings of M_1 and M_2 respectively. Furthermore, let R, S be closed surfaces with $R \subset \partial_- X_1 \subset \partial M_1$ and $S \subset \partial_- X_2 \subset \partial M_2$ and let $h : R \to S$ be a homeomorphism. Then the amalgamation of (X_1, Y_1) and (X_2, Y_2) along R, S via h is well defined.*

Proof: For the purposes of this proof, we must consider the decompositions (X_1, Y_1) and (X_2, Y_2) to be rigid decompositions (rather than being defined, merely, up to isotopy). Then, given these (rigid) Heegaard splittings (X_1, Y_1) and (X_2, Y_2) of M_1 and M_2 respectively, we choose (rigid) regular neighborhoods $N(R), N(S)$. Indeed, any two choices of regular neighborhoods of R, S are isotopic by the isotopy uniqueness of regular neighborhoods. (See [5].) But different (rigid) choices determine different sets of attaching disks. In the end, we need to establish that these (rigid) choices do not affect the outcome of the amalgamation (up to isotopy).

Denote a choice of regular neighborhood of R by N_r and the attaching disks for N_r in X_1 by $\mathcal{D}_r$. Denote a choice of regular neighborhood of S by N_s and the attaching disks for N_s in X_2 by $\mathcal{D}_s$. Our choices of regular neighborhood determine product structures. Denote the boundary component of N_r that contains the attaching disks $\mathcal{D}_r$, *i.e.*, the frontier of N_r, by $\partial_1 N_r$ and the frontier of N_s by by $\partial_1 N_s$.

A small isotopy ensures that near $\partial_1 N_r$, $(M_1 \backslash N_r) \cap X_1$ is a product $\mathcal{D}_r \times I$ (in an extension of the product structure defined by the regular neighborhood N_r). Likewise, a small isotopy ensures that near $\partial_1 N'_r$, $(M_1 \backslash N'_r) \cap X'_1$ is a product $\mathcal{D}'_r \times I$ (in an extension of the product structure defined by the regular neighborhood N'_r). Likewise, a small isotopy ensures that near $\partial_1 N_s$, $(M_2 \backslash N_s) \cap X_2$ is a product $\mathcal{D}_s \times I$ (in an extension of the product structure defined by the regular neighborhood N_s). Because they occur outside of N_r, N'_r and N_s, these small isotopies can also be performed after amalgamation, hence they do not affect the result of the amalgamation.

Our first task is to show that there is a choice of regular neighborhood $\tilde{N}_r$ of R with a single attaching disk per component of R, S and such that the result of amalgamation with choices N_r, N_s is isotopic to the result of amalgamation with choices $\tilde{N}_r, N_s$.

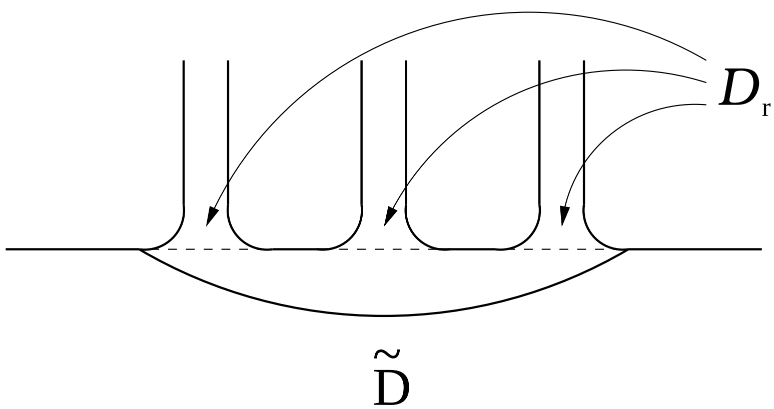

Figure 1: *Schematic rendition of $\tilde{D}$ after first isotopy*

Consider a component of $\partial_1 N_r$ and all components of $\mathcal{D}_r$ lying therein. Choose a circle c in this component of $\partial_1 N_r$ that bounds a disk $\tilde{D}$ (in this component of $\partial_1 N_r$) that contains all said components of $\mathcal{D}_r$. Furthermore, choose c so that $[\tilde{D}] \cap [D'] = \emptyset$ for all components D' of $\mathcal{D}_s$ and the equivalence relation given in the definition of amalgamation. Alter $\tilde{D}$ by a small isotopy that pushes the interior of $\tilde{D}$ into the interior of N_r. Abusing notation slightly, we continue to denote the disk resulting from this isotopy by $\tilde{D}$. See Figure 1.

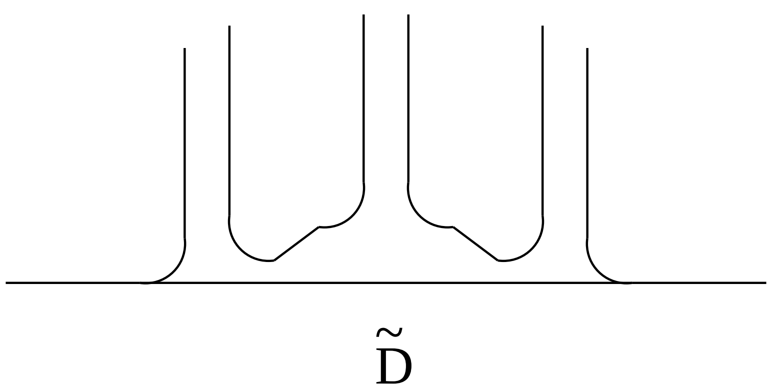

Figure 2: *Schematic rendition of $\tilde{D}$ after second isotopy*

Now consider an isotopy of $\tilde{D}$ that reverses the aforementioned isotopy, but that also isotopes the portion of X_1 lying above $\tilde{D}$ upwards. See Figure 2. We may assume that this isotopy is constant outside of a small neighborhood of $\tilde{D}$. We denote this isotopy by L.

Cutting X_1 along $\tilde{D}$ defines a new regular neighborhood of R. We denote the resulting regular neighborhood of R by $\tilde{N}_r$.

Claim 1: The result of amalgamation with choices N_r, N_s is isotopic to the result of amalgamation with choices $\tilde{N}_r, N_s$.

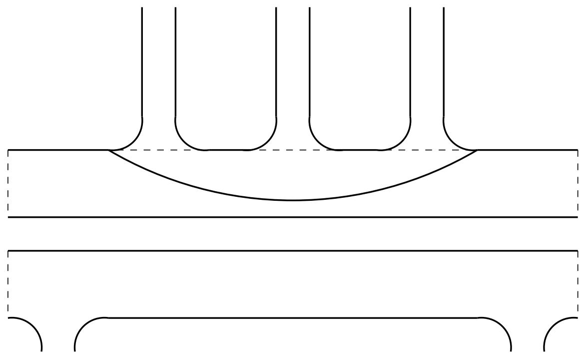

Figure 3: *Schematic rendition before amalgamation*

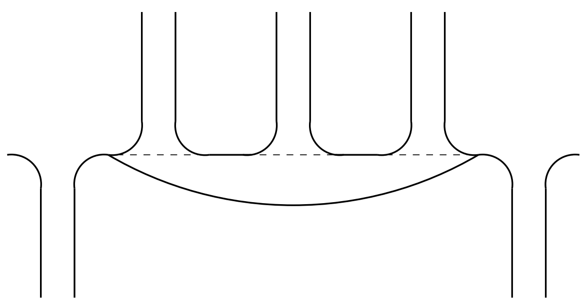

Figure 4: *Schematic rendition after amalgamation with choices N_r, N_s*

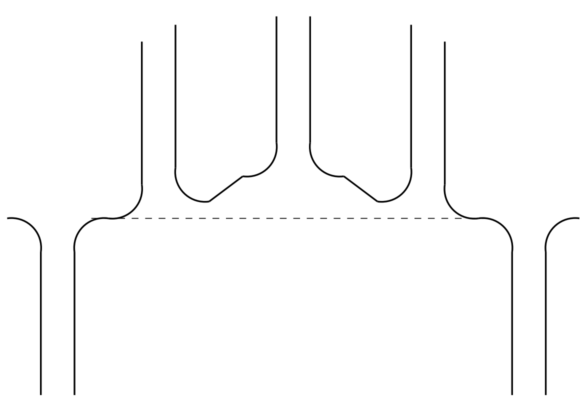

Figure 5: *Result of the isotopy L*

Since $[\tilde{D}] \cap [D'] = \emptyset$ for all components D' of $\mathcal{D}'_s$, a copy of $\tilde{D}$ (after the first isotopy) survives the amalgamation with choices N_r, N_s. See Figures 3 and 4.

Thus we may apply the isotopy L to obtain the result of amalgamation with choices $\tilde{N}_r, N_s$. See Figure 5. This proves Claim 1. //

A symmetric argument shows that given any choices of regular neighborhoods N_r, N_s of R, S, there are choices $\tilde{N}_r, \tilde{N}_s$ of R, S, the latter each with a single attaching disk per component, such that the result of amalgamation with choices N_r, N_s is isotopic to the result of amalgamation with choices $\tilde{N}_r, \tilde{N}_s$.

Claim 2: Suppose that $(X_1, Y_1), (X_1', Y_1')$ are isotopic (rigid) Heegaard splittings of M_1 with given choices of regular neighborhoods $N_r \subset X_1$, $N_r' \subset X_1'$. Further suppose that (X_2, Y_2) is a (rigid) Heegaard splitting of M_2 with given choice of regular neighborhood $N_s \subset X_2$. Then the result of amalgation of (X_1, Y_1) and (X_2, Y_2) is isotopic to the result of amalgamation of (X_1', Y_1') and (X_2, Y_2).

For ease of exposition, we will assume, as we may, that for each component of the regular neighborhoods of N_r, N_r' and N_s, there is only one attaching disk. Choose a component of N_r and denote it by $(N_r)_c$. Denote the corresponding component of N_r' by $(N_r')_c$ and that of N_s by $(N_s)_c$. Furthermore, denote the attaching disks for these components by D_r, D_r' and D_s, respectively. A regular neighborhood of M_1 in M intersects the amalgamations of the above Heegaard splittings in $(N_r \backslash (D_s \times I), Y_1 \cup (D_s \times I))$ and $(N_r' \backslash (D_s \times I), Y_1' \cup (D_s \times I))$, respectively. See Figures 6 and 7.

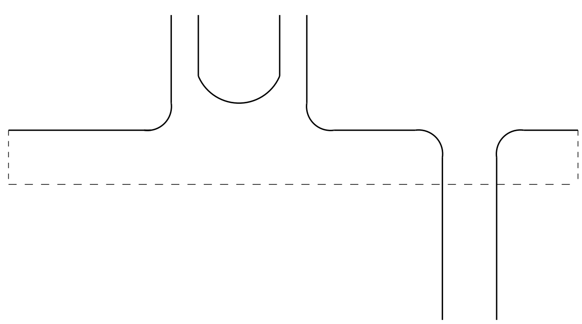

Figure 6: *The amalgamation of* (X_1, Y_1) *with* (X_2, Y_2)

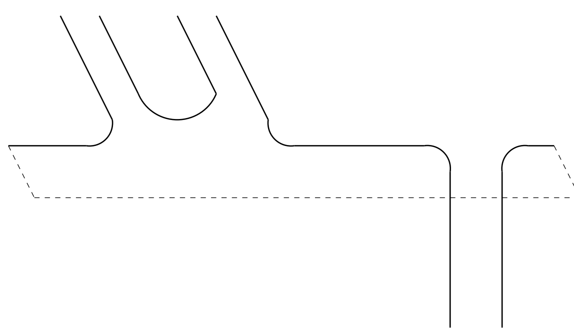

Figure 7: *The amalgamation of* (X_1, Y_1) *with* (X_2, Y_2) *after the isotopies*

Here M contains a copy of $M_1 \backslash N_r$, a shrunk version of M_1, that we denote by $\hat{M}_1$. By construction, $\hat{M}_1$ is homeomorphic to the copy of $M_1 \backslash N_r'$ contained in M. Hence we also denote the latter by $\hat{M}_1$. In M_1, Y_1 is isotopic to Y_1'. To establish Claim 2, it suffices to show that in the regular neighborhood of $\hat{M}_1$ in M, $Y_1 \cup (D_s \times I)$ is isotopic to $Y_1' \cup (D_s \times I)$. To this end, note that D_r, D_r' and D_s are regular neighborhoods of points r, r' and s, respectively. Thus D_r, D_r' and D_s can be shrunk arbitrarily small via isotopies. These isotopies can be performed either before or after the amalgamations and hence do not affect the outcome of the amalgamations.

The regular neighborhood of $\hat{M}_1$ in M contains a collar lying in $M \backslash \hat{M}_1$. We take the product structure on this collar determined by an extension of the product structure on N_s. Denote a vertical arc in this collar with endpoint s by α. An isotopy taking Y_1 to Y_1' may move s. Extend the isotopy of Y_1 over $\eta(\alpha)$ in such a way that the other endpoint of α is fixed. At the end of this isotopy, α is an arc in the collar with endpoints on opposite ends of the collar and can hence be isotoped to be vertical and thus to coincide with the original α. By transversality, r and s remain disjoint during these isotopies, hence so do D_r and D_s.

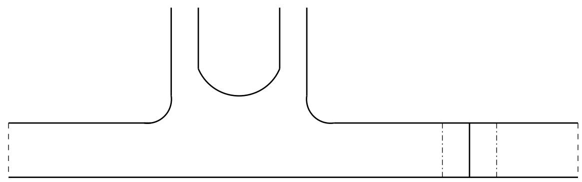

Figure 8: *The arc α before the isotopy*

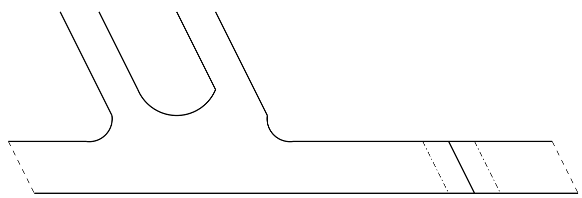

Figure 9: *The arc α after the first isotopy*

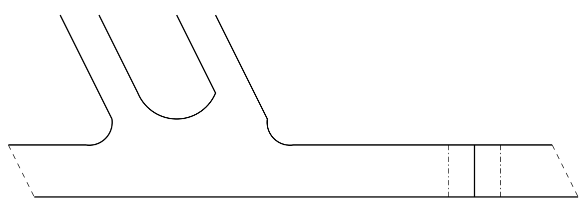

Figure 10: *The arc α after the second isotopy*

Amalgamating the resulting (rigid) Heegaard splittings at each stage of these two isotopies provides an isotopy between the result of amalgamating (X_1, Y_1) with (X_2, Y_2) and the result of amalgamating (X_1', Y_1') with (X_2, Y_2). This proves Claim 2. //

The proposition now follows from a symmetric argument for M_2. $\square$

References

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