Heegaard genus formula for Haken manifolds

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Abstract

Suppose M is a compact orientable 3-manifold and $Q \subset M$ a properly embedded orientable boundary incompressible essential surface. Denote the completions of the components of $M - Q$ with respect to the path metric by $M^1, \ldots, M^k$. Denote the smallest possible genus of a Heegaard splitting of M, or M^j respectively, for which ∂M, or ∂M^j respectively, is contained in one compression body by $g(M, \partial M)$, or $g(M^j, \partial M^j)$ respectively. Denote the maximal number of non parallel essential annuli that can be simultaneously embedded in M^j by n_j. Then

$$g(M, \partial M) \geq \frac{1}{5}(\sum_j g(M^j, \partial M^j) - |M - Q| + 5 - 2\chi(\partial_- V) + 4\chi(Q) - 4\sum_j n_j)$$

Heegaard splittings have long been used in the study of 3-manifolds. One reason for their continued importance in this study is that the Heegaard genus of a compact 3-manifold has proven to capture the topology of the 3-manifold more accurately than many other invariants. In particular, it provides an upper bound for the rank of the fundamental group of the 3-manifold, and this upper bound need not be sharp, as seen in the examples provided by M. Boileau and H. Zieschang in [1].

We here prove the following: Let M be a compact orientable 3-manifold and $Q \subset M$ an orientable boundary incompressible essential surface. Denote the completions of the components of $M - Q$ with respect to the path metric by $M^1, \ldots, M^k$. Denote the smallest possible genus of a Heegaard splitting of M, or M^j respectively, for which ∂M, or ∂M^j respectively, is contained in one compression body by $g(M, \partial M)$ or $g(M^j, \partial M^j)$ respectively. Here $g(M, \partial M)$ is called the <u>relative genus</u> of M. Denote the maximal number of non parallel essential annuli that can be simultaneously embedded in M^j by n_j. Then

$$g(M, \partial M) \geq \frac{1}{5}(\sum_j g(M^j, \partial M^j) - |M - Q| + 5 - 2\chi(\partial_- V) + 4\chi(Q) - 4\sum_j n_j)$$

A stronger inequality is obtained in the case in which M and the manifolds M^j are acylindrical.

The formula derived in this paper provides a topological analogue to the algebraic formula provided by R. Weidmann for the rank, i.e., the minimal number of generators, of a group (see [18]). He proves that if $G = A *_C B$ is a proper amalgamated product with malnormal amalgam C, then

$$\operatorname{rank} G \geq \frac{1}{3}(\operatorname{rank} A + \operatorname{rank} B - 2 \operatorname{rank} C + 5)$$

The group $C < G$ is <u>malnormal</u> if $gCg^{-1} \cap C = \{1\}$ for all $g \in G$. Suppose that M is a 3-manifold containing a separating incompressible surface Q and Q cuts M into two acylindrical 3-manifolds $M^1 \sqcup M^2$. Then Weidmann's formula tells us that the fundamental groups $\pi_1(M), \pi_1(M^1), \pi_1(M^2), \pi_1(Q)$ satisfy the following inequality:

$$r(\pi_1(M)) \geq \frac{1}{3}(r(\pi_1(M^1)) + r(\pi_1(M^2)) - 2r(\pi_1(Q)) + 5)$$

The correspondence between the two results makes the formula derived here particularly interesting, since it shows that the rank of the fundamental group and the genus of a 3-manifold satisfy a similar linear inequality. The construction and techniques used here are a generalization of those in joint work with M. Scharlemann [14]. The complexity here is considerably more substantial.

In his book [7], K. Johannson derives a variant of the Heegaard genus formula derived here for the special case in which M is closed and M^j is acylindrical (*i.e.*, it contains no essential annuli or tori) for $j = 1, \ldots, k$:

$$\sum_j g(M^j) \leq 5g(M) + 2g(Q)$$

Which is equivalent to the following:

$$g(M) \geq \frac{1}{5}(\sum_j g(M^j) - 2 + \chi(Q))$$

See [7, Proposition 23.40]. The inequality derived here applies in greater generality. In particular, Johannson's formula does not apply to the interesting case of a surface bundle over the circle.

Section 2 of this paper shows how the generalized Heegaard splitting of M induces generalized Heegaard splittings of the submanifolds M^j. This construction provides the generalized Heegaard splittings, but gives little control over their complexity. Section 3 provides more specifics on the construction in Section 2 that provide such control. Section 4 proves the Main Theorem. This proof consists entirely of adding up and subtracting the appropriate numbers from Sections 2 and 3.

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1 Preliminaries

For standard definitions concerning 3-manifolds, see [5] or [6].

Definition 1.1. *For L a properly embedded submanifold of M, we denote an open regular neighborhood of L in M by $\eta(L)$ and a closed regular neighborhood of L by $N(L)$. If L is an orientable surface properly embedded in M, then $N(L)$ is homeomorphic to $L \times [-1, 1]$. In this case we denote the subsets of $N(L)$ corresponding to $L \times [-1, 0]$ and $L \times [0, -1]$ by $N_l(L)$ and $N_r(L)$ respectively. We think of L itself as corresponding to $L \times \{0\}$.*

Definition 1.2. *A compression body is a 3-manifold W obtained from a closed orientable surface S by attaching 2-handles to $S \times \{-1\} \subset S \times I$ and capping off any resulting 2-sphere boundary components with 3-handles. We denote $S \times \{1\}$ by $\partial_+ W$ and $\partial W - \partial_+ W$ by $\partial_- W$.*

Dually, a compression body is an orientable 3-manifold obtained from a closed orientable surface $\partial_- W \times I$ or a $3-$ball or a union of the two by attaching 1-handles.

In the case where $\partial_- W = \emptyset$ (i.e., in the case where a $3-$ball was used in the dual construction of W), we also call W a handlebody. In the case where $W = \partial_+ W \times I$, we call W a trivial compression body.

In the case of a compression body that is not connected, we further require that all but one component of the compression body be a trivial compression body. The component of the compression body that is non trivial is called the active component.

In the following, we use the convention that $\chi(\emptyset) = 0$. Define the index of W by $J(W) = \chi(\partial_- W) - \chi(\partial_+ W)$.

The index will usually be a positive integer, but the index of a 3-ball is -2 and the index of a solid torus or of a trivial compression body is 0.

Definition 1.3. *A disk D that is properly embedded in a compression body W is essential if ∂D is an essential curve in $\partial_+ W$.*

Definition 1.4. *An annulus A in a compression body W is a spanning annulus if A is isotopic to an annulus of the form (simple closed curve)$\times I$ in the subset of W homeomorphic to $\partial_- W \times I$.*

Definition 1.5. *A set of defining disks for a compression body W is a set of disks $\{D_1, \ldots, D_n\}$ properly embedded in W with $\partial D_i \subset \partial_+ W$ for $i = 1, \ldots, n$ such that the result of cutting W along $D_1 \cup \cdots \cup D_n$ is homeomorphic to $\partial_- W \times I$ along with a collection of 3-balls.*

Definition 1.6. *A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V, W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$. We call S the splitting surface or Heegaard surface.*

If M is closed, the genus of M, denoted by $g(M)$, is the smallest possible genus of the splitting surface of a Heegaard splitting for M. If $\partial M \neq \emptyset$, then the relative genus of M, denoted by $g(M, \partial M)$, is the smallest possible genus of the splitting surface of a Heegaard splitting for which ∂M is entirely contained in one of the compression bodies.

Definition 1.7. *A Heegaard splitting $M = V \cup_S W$ is reducible if there are essential disks $D_1 \subset V$ and $D_2 \subset W$, such that $\partial D_1 = \partial D_2$. A Heegaard splitting which is not reducible is irreducible.*

A Heegaard splitting $M = V \cup_S W$ is weakly reducible if there are essential disks $D_1 \subset V$ and $D_2 \subset W$, such that $\partial D_1 \cap \partial D_2 = \emptyset$. A Heegaard splitting which is not weakly reducible is strongly irreducible.

Definition 1.8. *Let $M = V \cup_S W$ be a Heegaard splitting. A Heegaard splitting is stabilized if there are disks $D \subset V, E \subset W$ such that $|\partial D \cap \partial E| = 1$. A destabilization of $M = V \cup_S W$ is a Heegaard splitting obtained from $M = V \cup_S W$ by cutting along the cocore of a 1-handle. (E.g., if $D \subset V, E \subset W$ is a stabilizing pair of disks, then D is the cocore of a 1-handle of V and the existence of E guarantees that the result of cutting along D results in a Heegaard splitting.) We say that a Heegaard splitting $M = X \cup_T Y$ is a stabilization of $M = V \cup_S W$, if there is a sequence of Heegaard splittings $M = X^1 \cup_{T^1} Y^1, \ldots, M = X^l \cup_{T^l} Y^l$ with $X^1 = X, Y^1 = Y, X^l = V, Y^l = W$ and $X^r \cup_{T^r} Y^r$ is obtained from $X^{r-1} \cup_{T^{r-1}} Y^{r-1}$ by a destabilization.*

The notion of strong irreducibility, due to Casson and Gordon in [3], prompted the following definition due to Scharlemann and Thompson.

Definition 1.9. *A* <u>*generalized Heegaard splitting*</u> *of a compact orientable* 3*-manifold* M *is a decomposition* $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$*. Each of the* V_i *and* W_i *is a compression body,* $\partial_+ V_i = S_i = \partial_+ W_i$*, (i.e.,* $V_i \cup_{S_i} W_i$ *is a Heegaard splitting of a submanifold of* M*) and* $\partial_- W_i = F_i = \partial_- V_{i+1}$*. We say that a generalized Heegaard splitting is* <u>*strongly irreducible*</u> *if each Heegaard splitting* $V_i \cup_{S_i} W_i$ *is strongly irreducible and each* F_i *is incompressible in* M*. We will denote* $\cup_i F_i$ *by* $\mathcal{F}$ *and* $\cup_i S_i$ *by* $\mathcal{S}$*. The surfaces in* $\mathcal{F}$ *are called the* <u>*thin levels*</u> *and the surfaces in* $\mathcal{S}$ *the* <u>*thick levels*</u>*.*

Let $M = V \cup_S W$ *be an irreducible Heegaard splitting. We may think of* M *as being obtained from* $\partial_- V \times I$ *by attaching all* 1*-handles in* V *followed by all* 2*-handles in* W*, followed, perhaps, by* 3*-handles. An* <u>*untelescoping*</u> *of* $M = V \cup_S W$ *is a rearrangement of the order in which the* 1*-handles of* V *and the* 2*-handles of* W *are attached. This rearrangement yields a generalized Heegaard splitting. If the untelescoping is strongly irreducible, then it is called a* <u>*weak reduction*</u> *of* $M = V \cup_S W$*. Here* $\partial_- V_1 = \partial_- V$*. For convenience, we will occasionally denote* $\partial_- V_1$ *by* F_0 *and* $\partial_- W_n$ *by* F_n*.*

The Main Theorem in [11] together with the calculation [13, Lemma 2] implies the following:

Theorem 1.10. *Suppose* M *is an irreducible compact* 3*-manifold. Then* M *possesses an unstabilized genus* g *Heegaard splitting*

$$M = V \cup_S W$$

if and only if M *has a strongly irreducible generalized Heegaard splitting*

$$M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$$

with $\partial_- V_1 = \partial_- V$ *such that*

$$\sum_{i=1}^{m} J(V_i) = 2g - 2 + \chi(\partial_- V).$$

The details can be found in [10]. Roughly speaking, one implication comes from taking a weak reduction of a given Heegaard splitting of genus g, the other from thinking of a given generalized Heegaard splitting as a weak reduction of some Heegaard splitting. The latter process is called the *amalgamation* (for details see [15]) of the generalized Heegaard splitting.

A strongly irreducible Heegaard splitting can be isotoped so that its splitting surface, S, intersects an incompressible surface, P, only in curves essential in both S and P. This is a deep fact and is proven, for instance, in [16, Lemma 6]. This fact, together with the fact that incompressible surfaces can be isotoped to meet only in essential curves, establishes the following:

Lemma 1.11. *Let* P *be a properly embedded incompressible surface in an irreducible* 3*-manifold* M *and let* $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ *be a strongly irreducible generalized Heegaard splitting of* M*. Then* $\mathcal{F} \cup \mathcal{S}$ *can be isotoped to intersect* P *only in curves that are essential in both* P *and* $\mathcal{F} \cup \mathcal{S}$*.*

2 The Construction

In this section, we suppose that M is a compact orientable 3-manifold with generalized Heegaard splitting $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ and Q a compact orientable surface in M with $\partial Q \subset \partial M = \partial_- V = \partial_- V_1$. Assuming that Q can be isotoped so that all components of $Q \cap (\mathcal{S} \cup \mathcal{F})$ are essential in both Q and $\mathcal{S} \cup \mathcal{F}$, we describe a construction for generalized Heegaard splittings of the completion of the components of $M - Q$ with respect to the path metric.

In simpler contexts, a more concrete approach to this sort of construction has been used. A rough sketch is as follows: Consider a Heegaard splitting $M = V \cup_S W$. Make V very thin, so it intersects Q in a collar of ∂Q along with disks. Then cut along Q and consider the completion of a component C of $M - Q$. Add a collar of the copies of components of Q in the boundary of C to $V \cap C$. It is a nontrivial fact that this would indeed yield a Heegaard splitting of C and is the basis for the construction in [7]. However, the approach there, though simpler at the outset, requires far more work to gain control over the number of components in $V \cap Q$. Thus, although the construction in this section appears to be more complicated than required, it will become evident in the following sections that it allows for more satisfactory control over the sum of indices of the resulting generalized Heegaard splittings.

In the following, we will abuse notation slightly and consider $Q \times [-1, 1]$ to be lying in M with $Q = Q \times \{0\}$ via the homeomorphisn with $N(Q)$. We will further assume that $F_i \cap (Q \times [-1, 1]) = (F_i \cap Q) \times [-1, 1]$ and similarly for S_i, for all i. Denote the completions of the components of $M - Q$ with respect to the path metric, *i.e.*, the 3-manifolds into which Q cuts M, by $M^1, \dots, M^k$. Note that $M^1 \sqcup \dots \sqcup M^k$ is homeomorphic to $M - (Q \times (-1, 1))$.

Definition 2.1. *A properly embedded surface Q in a 3-manifold M is essential if it is incompressible and not boundary parallel.*

Remark 2.2. *An essential surface can be boundary compressible. Recall that if a surface Q in a 3-manifold M is boundary compressible, then there is a disk D in M such that interior$(D) \cap Q = \emptyset$ and such that boundary $\partial D = a \cup b$ with a, b connected arcs and $a \subset Q$ and $b \subset \partial M$. Supposing that Q is boundary compressible in M, then D provides instructions for modifying Q. Specifically, replace a small collar of a in Q by two parallel copies of D. This modification is called a boundary compression of Q along D. Here D is called a boundary compressing disk for Q.*

Definition 2.3. *The two copies of Q in $\partial(M^1 \sqcup \dots \sqcup M^k)$ are called the remnants of Q.*

Definition 2.4. *Let F be a closed orientable surface. A generalized compression body is an orientable 3-manifold W obtained from $F \times I$ or a 3-ball or a union of the two by attaching 1-handles. If attached to $F \times I$, the 1-handles must be attached to $F \times \{1\}$.*

We denote $F \times \{-1\}$ by $\partial_- W$. We denote $\partial F \times I$ by $\partial_v W$ and $\partial W - (\partial_- W \cup \partial_v W)$ by $\partial_+ W$.

A set of defining disks for W is a set of disks $\mathcal{D}$ with boundary in $\partial_+ W$ that cut W into $\partial_- W \times I$ together with a collection of 3-balls.

Lemma 2.5. *Suppose that W is a generalized compression body and $Q \subset W$ is a properly embedded connected incompressible surface disjoint from $\partial_v W$. Suppose further that Q meets $\partial_+ W$ and that $\chi(Q) \leq 0$. Then either Q is a spanning annulus, or there is a boundary compressing disk D for Q such that $\partial D \cap \partial W \subset \partial_+ W$.*

Proof: Let $\mathcal{D}$ be a set of defining disks for W. Since Q is incompressible, an innermost disk argument shows that Q can be isotoped so that it intersects the components of $\mathcal{D}$ in arcs. Furthermore, an outermost arc argument shows that, after isotopy, any such arc of intersection is essential in Q. Now if there are arcs of intersection, then we choose one that is outermost in $\mathcal{D}$ and see that the outermost disk it cuts off is a boundary compressing disk for Q.

If there are no such arcs of intersection, then we cut along $\mathcal{D}$ to obtain a 3-manifold homeomorphic to $\partial_- W \times I$. It is well known that an incompressible and boundary incompressible surface in a product is either horizontal or vertical. Here the horizontal case is ruled out because Q does not meet $\partial_v W$. Thus Q is either a spanning annulus or boundary compressible.

Now suppose that Q is boundary compressible via a disk $\tilde{D}$ such that $\partial \tilde{D} \cap \partial W \subset \partial_- W$. Then since ∂Q meets $\partial_+ W$ there is a dual boundary compressing disk D such that $\partial D \cap \partial W \subset \partial_+ W$, as required. □

Remark 2.6. *The construction here is relevant in the case in which for each component Q_c of Q, $Q_c \cap (\mathcal{F} \cup \mathcal{S}) \neq \emptyset$. If, on the other hand, there is a component Q_c of Q for which $Q_c \cap (\mathcal{F} \cup \mathcal{S}) = \emptyset$, then we may treat this component separately. As it lies entirely in one of the compression bodies $V_1, W_1, \ldots, V_n, W_n$, it must in fact be parallel to a component of $\mathcal{F}$. If components of Q are parallel into components of $\mathcal{F}$, then a much simpler construction yields a stronger result, see Proposition 4.1.*

Lemma 2.7. *Suppose $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a generalized Heegaard splitting and suppose $Q \subset M$ is an essential boundary incompressible surface. Also suppose that no component of Q is parallel into $\mathcal{F}$. Suppose furthermore that Q has been isotoped so that all components of $Q \cap (\mathcal{S} \cup \mathcal{F})$ are essential in both Q and $\mathcal{S} \cup \mathcal{F}$ and so that the number of components in this intersection is minimal subject to this condition. Then for each i, each component of the completion of $V_i - Q$ and $W_i - Q$ with respect to the path metric is a generalized compression body.*

Proof: Note that under the above assumptions there will be no component of $Q \cap V_i$ that does not meet $\partial_+ V_i$, for such a component would be parallel into $\partial_- V_i$. Thus each component of $Q \cap V_i$ satisfies the hypotheses of Lemma 2.5. Similarly for $Q \cap W_i$.

Let $\tilde{Q}$ be a component of $Q \cap V_i$. Since each component of $\tilde{Q} \cap \partial_+ V_i \subset Q \cap \partial_+ V_i$ is essential in $\partial_+ V_i$ and in Q, $\tilde{Q}$ is not a disk. If $\tilde{Q}$ is a spanning annulus, then we may cut along this spanning annulus and obtain a generalized compression body. If $\tilde{Q}$ is boundary compressible via a boundary compressing disk that meets $\partial_+ V_i$, then we may perform the boundary compression along this disk to obtain $\tilde{Q}_b$. The components of the completion of $V_i - \tilde{Q}$ with respect to the path metric can be obtained from the components of the completion of $V_i - \tilde{Q}_b$ with respect to the path metric by attaching a 1-handle with cocore the boundary compressing disk.

We prove the lemma by induction on $-\chi(Q \cap V_i)$. This is accomplished by repeated application of the argument above. The same holds for $Q \cap W_i$. □

Definition 2.8. *For a submanifold $N \subset M$, we will denote $N \cap M^j$ by N^j. E.g., $S_3^1 = S_3 \cap M^1$, $W_5^2 = W_5 \cap M^2$, $Q^j = Q \cap M^j$.*

Note that S_i^j and F_i^j will typically not be closed surfaces. Also, V_i^j and W_i^j will typically not be compression bodies, only generalized compression bodies. The following construction appears to be a fairly natural way of "capping off" the components of $\partial_v V_i^j$ and $\partial_v W_i^j$ with appropriate ((punctured surface) $\times I$)'s. This is the first step in constructing generalized Heegaard splittings on the submanifolds M^j. The difficulty lies in "capping off" $\partial_v V_i^j$ and $\partial_v W_i^j$ in a way that is consistent.

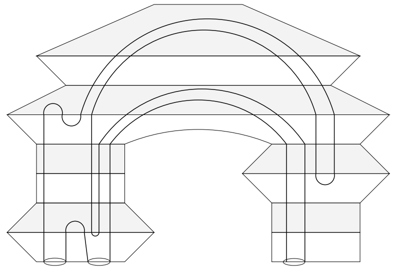

Figure 1: The surface Q in the generalized Heegaard splitting

Construction 2.9. *(The Main Construction) Let M be a compact possibly closed orientable irreducible 3-manifold. Let*

$$M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$$

be a strongly irreducible generalized Heegaard splitting of M. Let Q be a compact possible closed orientable essential boundary incompressible not necessarily connected surface properly embedded in M. Denote $\partial_- V_1$ by $\partial_- M$ and $\partial_- W_n$ by $\partial_+ M$. Then $\partial M = \partial_- M \cup \partial_+ M$. Suppose that $\partial Q \subset \partial_- M$. Suppose further that no component of Q is parallel to a component of $\mathcal{F}$ and that $\mathcal{S} \cup \mathcal{F}$ has been isotoped so that all components of $Q \cap (\mathcal{S} \cup \mathcal{F})$ are essential in both Q and $\mathcal{S} \cup \mathcal{F}$ and so that the number of such components of intersection is minimal.

Denote the completions of the components of $M - Q$ with respect to the path metric by $M^1, \ldots, M^k$. We construct generalized Heegaard splittings for $M^1, \ldots, M^k$, respectively, from

$$M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$$

We call these generalized Heegaard splittings the induced Heegaard splittings of $M^1, \ldots, M^k$, respectively.

Let $h : M \to [0, 1]$ be a Morse function corresponding to $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ with $S_i = h^{-1}(s_i)$ and $F_i = h^{-1}(f_i)$ for appropriate $s_1, \ldots, s_n$ and $f_1, \ldots, f_{n-1}$. Note that $s_1, \ldots, s_n, f_1, \ldots, f_{n-1}$ are regular values of h. Our assumptions on $\mathcal{S} \cup \mathcal{F}$ guarantee that $s_1, \ldots, s_n, f_1, \ldots, f_{n-1}$ are also regular values of $h|_Q$.

Set $q_t = Q \cap h^{-1}(t)$ and $Q_{[t_1,t_2]} = Q \cap h^{-1}([t_1, t_2])$. Here q_t will consist of a collection of circles (at least for regular values of $h|_Q$) and $Q_{[t_1,t_2]}$ will be a subsurface of Q with $\partial Q_{[t_1,t_2]} = q_{t_1} \cup q_{t_2}$. Consider a bicollar $Q \times [-1, 1]$ of Q in M. If $q \in (Q \cap \mathcal{S} \cup \mathcal{F})$, then we assume that $h(q, t) = h(q)$ for all $t \in I$.

For all i, set

$$Q^-_{s_i} = (q_{s_i} \times [-1, -1 + \frac{1}{2i}]) \cup (Q_{[s_i,1]} \times \{-1 + \frac{1}{2i}\})$$

$$Q^+_{s_i} = (q_{s_i} \times [1 - \frac{1}{2i}, 1]) \cup (Q_{[s_i,1]} \times \{1 - \frac{1}{2i}\})$$

For all $0 < i < n$, set

$$Q^-_{f_i} = (q_{f_i} \times [-1, -1 + \frac{1}{2i+1}]) \cup (Q_{[f_i,1]} \times \{-1 + \frac{1}{2i+1}\})$$

$$Q^+_{f_i} = (q_{f_i} \times [1 - \frac{1}{2i+1}, 1]) \cup (Q_{[f_i,1]} \times \{1 - \frac{1}{2i+1}\})$$

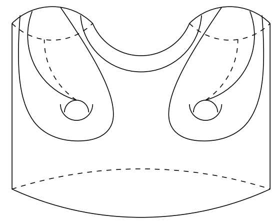

Figure 2: A collection of arcs α_i that cuts $Q_{[f_{i-1},s_i]}$ into a spanning annulus

Recall that $F_0 = \partial_- V_1$ and $F_n = \partial_- W_n$. Set

$$Q^-_{f_0} = (q_{f_0} \times [-1, -\frac{1}{100}]) \cup (Q \times \{-\frac{1}{100}\})$$

$$Q^+_{f_0} = (q_{f_0} \times [\frac{1}{100}, 1]) \cup (Q \times \{\frac{1}{100}\})$$

$$Q^{\pm}_{f_n} = \emptyset$$

For all i, set

$$\tilde{F}_i = (F_i - (F_i \cap (Q \times [-1,1]))) \cup Q^+_{f_i} \cup Q^-_{f_i}$$

and

$$\tilde{S}_i = (S_i - (S_i \cap (Q \times [-1,1]))) \cup Q^+_{s_i} \cup Q^-_{s_i}$$

Then for all i, $\tilde{F}_i$ and $\tilde{S}_i$ are closed surfaces. Note in particular that since F_n does not meet Q, we have $\tilde{F}_n = F_n$. Let $\tilde{V}_i$ be the cobordism between $\tilde{F}_{i-1}$ and $\tilde{S}_i$ for $i = 1, \ldots, n$ and let $\tilde{W}_i$ be the cobordism between $\tilde{S}_i$ and $\tilde{F}_i$ for $i = 1, \ldots, n$. Here neither $\tilde{V}_i^j$ nor $\tilde{W}_i^j$ need be a compression body.

Let α_i be a union of properly embedded arcs in $Q_{[f_{i-1},s_i]}$ disjoint from $q_{f_{i-1}}$ that cut $Q_{[f_{i-1},s_i]}$ into disks and spanning annuli, see Figure 2. I.e., $Q_{[f_{i-1},s_i]} - \alpha_i$ is homeomorphic to $(q_{f_{i-1}} \times I) \cup$ (disks). Analogously, choose β_i in $Q_{[s_i,f_i]}$. Do this in such a way that $\partial\alpha_i \cap \partial\beta_i = \emptyset$. Then, set $\dot{V}_i^j = (\tilde{V}_i^j - (\eta(\alpha_i \times \{\pm 1\})^j) \cup (\eta(\beta_i \times \{\pm 1\})^j)$ and $\dot{W}_i^j = (\tilde{W}_i^j - (\eta(\beta_i \times \{\pm 1\})^j) \cup (\eta(\alpha_i \times \{\pm 1\})^j)$.

Claim 2.10. *$\dot{V}_i^j$ and $\dot{W}_i^j$ are compression bodies for $i = 1, \ldots, n$ and $j = 1, \ldots, k$.*

Consider the construction of $\dot{V}_i^j$. Since each component of $Q \cap (\mathcal{S} \cup \mathcal{F})$ is essential in both Q and $\mathcal{S} \cup \mathcal{F}$ and since the number of such components is minimal, $Q_{[f_{i-1},s_i]} = Q \cap V_i$ is an essential surface. Note that $Q_{[f_{i-1},s_i]}$ might be boundary compressible, but that the minimality assumption on the number of components of $Q \cap (\mathcal{S} \cup \mathcal{F})$ guarantees that the hypotheses of Lemma 2.7 are met. Hence V_i^j is a generalized compression body. Here $\tilde{V}_i^j$ is obtained from V_i^j by adjoining a 3-manifold homeomorphic to $Q_{[f_{i-1},1]} \times I$ along a subsurface homeomorphic to $Q_{[f_{i-1},s_i]}$. Furthermore $\dot{V}_i^j$ is obtained from V_i^j by adjoining the same 3-manifold but along $\partial_v V_i^j = (q_{f_i} \times \{\pm 1\}) \times I$ and disks and attaching 1-handles. The result is thus a compression body. Similarly for $\dot{W}_i^j$.

Set $\dot{S}_i^j = \partial_+ \dot{V}_i^j = \partial_+ \dot{W}_i^j$ and $\dot{F}_i^j = \partial_- \dot{W}_i^j = \partial_- \dot{V}_{i+1}^j$ for all i, j. Further set $\dot{F}_0^j = \partial_- \dot{V}_1^j$ and $\dot{F}_n^j = \partial_- \dot{W}_n^j$ for all j. The generalized Heegaard splitting induced on M^j is $M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j)$.

Strictly speaking, the compression bodies may have to be relabelled. For recall that if a compression body is not connected, then it has exactly one active component. Suppose, for instance, that $\dot{V}_1^1$ is not connected and that more than one component is non trivial. (Note that it follows that $\dot{W}_1^1$ is also not connected. For $|\dot{V}_1^1| = |\partial_+\dot{V}_1^1| = |\partial_+\dot{W}_1^1| = |\dot{W}_1^1|$.) We then insert trivial compression bodies and relabel as necessary.

Remark 2.11. *Despite the possible insertion of trivial compression bodies and relabelling, we maintain the notation* $M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j)$. *This helps us to keep track of the relation to the original generalized Heegaard splitting on* M. *This is especially helpful in the computations below.*

Lemma 2.12. *(Preliminary Count) For the construction above,*

$$\sum_j \sum_i J(\dot{V}_i^j) = \sum_i (J(V_i) + 2\chi(Q_{[f_{i-1},s_i]}) + 4(|\alpha_i| + |\beta_i|))$$

Proof: Consider the Euler characteristics of the surfaces in the construction above. Since the Euler characteristic of a circle is 0, $\sum_j \chi(S_i^j) = \chi(S_i)$, for $i = 1, \ldots, n$ and $\sum_j \chi(F_i^j) = \chi(F_i)$, for $i = 0, \ldots, n-1$. Furthermore, $\sum_j \chi(\tilde{S}_i^j) = \sum_j \chi(S_i^j) + 2\chi(Q_{[s_i,1]})$ and $\sum_j \chi(\tilde{F}_i^j) = \sum_j \chi(F_i^j) + 2\chi(Q_{[f_i,1]})$. Thus $\sum_j \chi(\dot{S}_i^j) = \sum_j \chi(S_i^j) + 2\chi(Q_{[s_i,1]}) - 4|\alpha_i| - 4|\beta_i|$, and $\sum_j \chi(\dot{F}_i^j) = \sum_j \chi(F_i^j) + 2\chi(Q_{[f_i,1]})$. Hence

$$\sum_i \sum_j J(\dot{V}_i^j) =$$

$$\sum_i \sum_j (\chi(\dot{F}_{i-1}^j) - \chi(\dot{S}_i^j)) =$$

$$\sum_i \sum_j (\chi(F_{i-1}^j) - \chi(S_i^j)) + 2(\chi(Q_{[f_{i-1},1]}) - \chi(Q_{[s_i,1]}) + 4(|\alpha_i| + |\beta_i|)) =$$

$$\sum_i (\chi(F_{i-1}) - \chi(S_i) + 2\chi(Q_{[f_{i-1},s_i]}) + 4(|\alpha_i| + |\beta_i|)) =$$

$$\sum_i (J(V_i) + 2\chi(Q_{[f_{i-1},s_i]}) + 4(|\alpha_i| + |\beta_i|))$$

□

3 How many arcs do we need?

In order to perform the required calculations we must count the number of arcs required for α_i, β_i in the Main Construction. To do so, we define α_i, β_i more systematically.

Lemma 3.1. *In the Main Construction we may choose* α_i *so that the number of components of* α_i *is equal to* $-\chi(Q_{[f_{i-1},s_i]}) + d_i$, *where* d_i *is the number of components of* $Q_{[f_{i-1},s_i]}$ *that do not meet* F_{i-1}.

Proof: Recall that we are assuming that $\mathcal{S} \cup \mathcal{F}$ has been isotoped so that all components of $Q \cap (\mathcal{S} \cup \mathcal{F})$ are essential in both Q and $\mathcal{S} \cup \mathcal{F}$ and so that the number of such components of intersection is minimal.

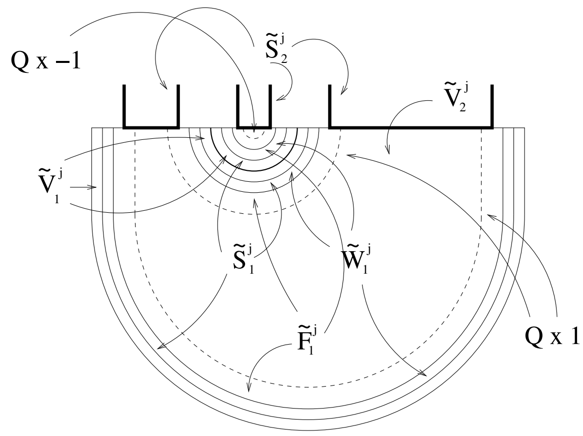

Figure 3: The construction of $\tilde{V}_2^j$

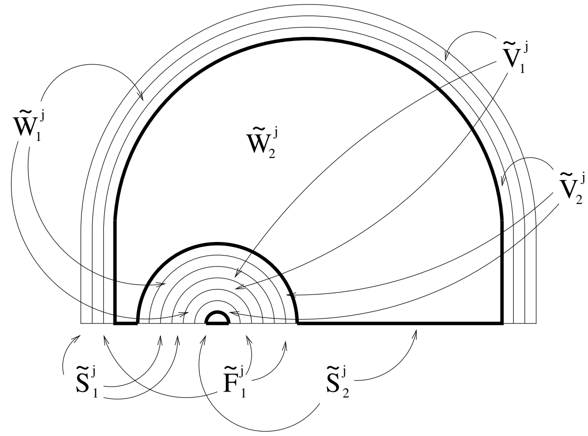

Figure 4: The construction of $\tilde{W}_2^j$

We proceed by induction on $-\chi(Q_{[f_{i-1},s_i]})$. If $-\chi(Q_{[f_{i-1},s_i]}) = 0$, then $Q_{[f_{i-1},s_i]}$ consists of annuli. If a component $\tilde{A}$ of $Q_{[f_{i-1},s_i]}$ is not a spanning annulus, then the above minimality assumption implies that its boundary lies in $\partial_+ V_i$. Thus Lemma 2.5 locates a boundary compressing disk D for $\tilde{A}$ such that $\partial D = a \cup b$ with $a \subset \tilde{A}$ and $b \subset \partial_+ V_i$. In this case α_i consists of the arcs a for each such annulus. Take α^0 to be the set of such spanning arcs.

Suppose $-\chi(Q_{[f_{i-1},s_i]}) > 0$ and let $\tilde{Q}$ be a component of $Q_{[f_{i-1},s_i]}$ for which $-\chi(\tilde{Q}) > 0$. Again, Lemma 2.5 locates a boundary compressing disk D for $\tilde{Q}$ such that $\partial D = a \cup b$ with $a \subset \tilde{Q}$ and $b \subset \partial_+ V_i$. Set $\alpha^n = a$.

Consider the surface Q^1 obtained from $Q_{[f_{i-1},s_i]}$ by a boundary compression along D. Then $-\chi(Q^1) = -\chi(Q_{[f_{i-1},s_i]}) - 1$. By inductive hypothesis there is a collection of arcs $\alpha^0, \dots, \alpha^{n-1}$ such that the complement of $\alpha^0 \cup \dots \cup \alpha^{n-1}$ in Q^1 consists of spanning annuli and disks. Now take α_i to be the collection $\alpha^0, \dots, \alpha^n$.

The fact that the number of components of α_i is equal to $-\chi(Q_{[f_{i-1},s_i]}) + d_i$ follows from our choice of α_i as a subcollection of the arcs in α_1 that cuts $Q_{[f_{i-1},s_i]}$ into spanning annuli and disks and that has the minimal number of components among all such subcollections. $\square$

The same strategy could be used to locate a collection of arcs β_i. But there is a crucial difference between $Q_{[f_{i-1},s_i]}$ and $Q_{[s_i,f_i]}$. This asymmetry in our construction may be exploited to show that in fact all arcs in β_i are superfluous. See Figures 3 and 4.

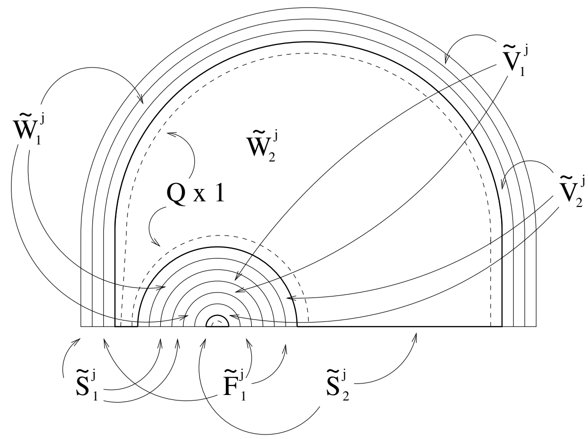

Figure 5: A boundary compressing disk intersecting $Q_{[s_2,f_2]} \times I \subset Q \times I$

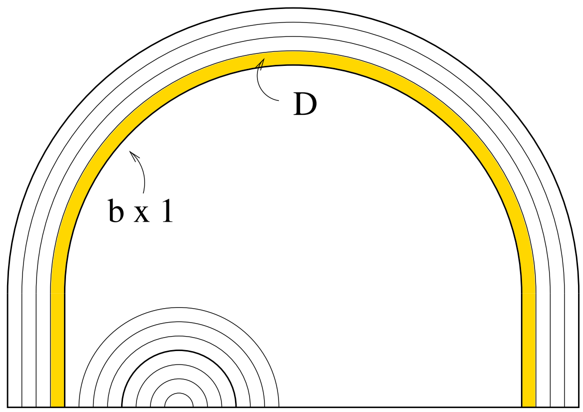

Figure 6: A destabilizing disk for the component b of β_2

Lemma 3.2. *In the Main Construction, we may choose,* $\beta_i = \emptyset$.

Proof: Consider a collection of arcs β_i in $Q_{[s_i,f_i]}$ constructed via the argument in Lemma 3.1. Observe that each arc found in Lemma 3.1 was part of the boundary of a boundary compressing disk $\tilde{D}$. Here $\partial\tilde{D} = b \cup c$ with $b \in Q_{[s_i,f_i]}$ and $c \in \partial_+ W_i$. Denote the collection of boundary compressing disks corresponding to the components of β_i by $\mathcal{D}_{\beta_i}$. We may choose $\mathcal{D}_{\beta_i}$ so that its components are pairwise disjoint. Note however, that a component of $\mathcal{D}_{\beta_i}$ may have to intersect $Q_{[s_i,f_i]}$ in its interior.

Let D be a component of $\mathcal{D}_{\beta_i}$ and consider how D meets $\tilde{\mathcal{F}} \cup \tilde{\mathcal{S}}$. See Figure 5 for the case $i = 2$. Then the corresponding arc, call it $b \times 1$, in β_i is parallel to $\tilde{S}_i^j$ via the disk sketched in Figure 6. Extending D into the other side of $Q_{[s_i,f_i]}$ allows us to locate another such disk for $b \times -1$.

The cocore of the 1-handle attached along the arc and a truncated version of this disk define a destabilizing pair. In this way, each component of β_i is in fact superfluous. $\square$

The procedures above show that the number of components of $Q_{[f_{i-1},s_i]}$ that do not meet $q_{f_{i-1}}$ plays a role in the complexities of the generalized Heegaard splittings constructed. It is of particular importance to control the contribution arising from annular components of this type.

Definition 3.3. *An annulus A in a compression body W is called a* <u>*dipping annulus*</u> *if it is essential and* $\partial A \subset \partial_+ W$.

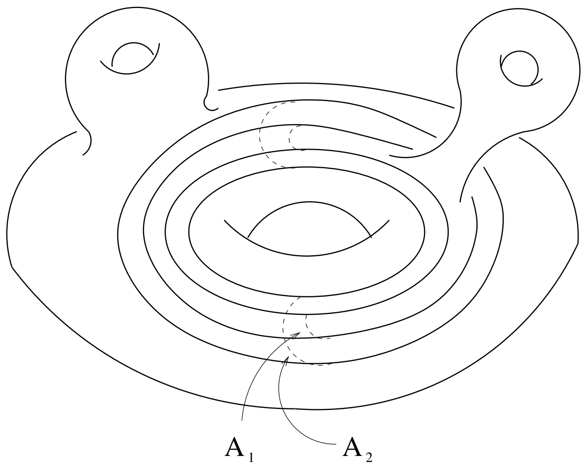

Figure 7: Isotopic dipping annuli

Lemma 3.4. *Suppose $\mathcal{B}$ is a collection of essential annuli in a compression body W and that $\mathcal{A}$ is the subcollection consisting of dipping annuli. Denote the number of annular components of $\partial_+ W - \mathcal{B}$ by l. If no two components of $\mathcal{A}$ are isotopic, then*

$$|\mathcal{A}| \leq J(W) + \frac{l}{2}$$

Proof: This is [14, Lemma 7.3]. □

Corollary 3.5. *Suppose $\mathcal{B}$ is a collection of essential surfaces in a compression body W and that $\mathcal{A}$ is the subcollection consisting of dipping annuli. Denote the number of annular components of $\partial_+ W - \mathcal{B}$ that meet dipping annuli by l. Then*

$$|\mathcal{A}| \leq J(W) + \frac{l}{2}$$

Proof: First note that the conclusion depends only on $\mathcal{A}$, so we may ignore $\mathcal{B} - \mathcal{A}$ in the proof. The proof is by induction on the number of parallel dipping annuli. If this number is 0, then the conclusion follows from Lemma 3.4. To verify the inductive step, observe that deleting a dipping annulus that is parallel to another decreases the number of annular components of $\partial_+ W - \mathcal{B}$ by two. □

Definition 3.6. *Denote by n_j the maximal number of pairwise non isotopic annuli that can be simultaneously embedded in M^j.*

Recall also that d_i is the number of components of $Q_{[f_{i-1},s_i]}$ that do not meet F_{i-1}.

Lemma 3.7. *In the counting arguments below, we may assume that in Lemma 3.1*

$$\sum_i d_i \leq \sum_i (-\chi(Q_{[f_{i-1},s_i]}) + J(V_i)) + 2\sum_j n_j$$

Proof: Recall that d_i is the number of components of $Q_{[f_{i-1},s_i]}$ that do not meet $\partial_- V_i$. Denote the number of components of $Q_{[f_{i-1},s_i]}$ that are dipping annuli by a_i. Then

$$d_i \leq a_i - \chi(Q_{[f_{i-1},s_i]})$$

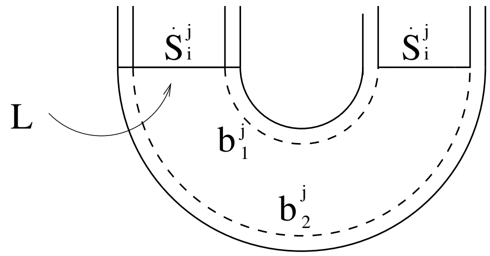

Figure 8: The annulus L

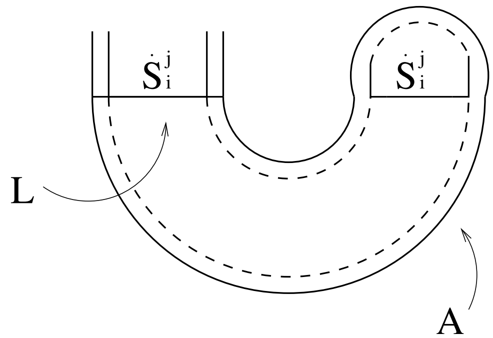

Figure 9: The annuli L and A

Corollary 3.5 provides a bound on a_i, but this bound depends on the number of annular components of $S_i - Q$ that meet dipping annuli. See for instance Figure 7. This number is potentially unbounded, but we will show below that corresponding to each annular component in $S_i - Q$ there is a destabilization of $\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j$.

So consider an annulus L in $\cup_i S_i - Q$ that meets at least one dipping annulus. See Figure 8.

Case I: L is isotopic into ∂M^j.

In this case there is an annulus A in ∂M^j as pictured in Figure 9.

Since L is boundary parallel, M^j is in fact homeomorphic to one of the two components C_1 or C_2, say C_1, obtained by taking the completion with respect to the path metric of $M^j - L$. There is thus a simpler generalized Heegaard splitting for M^j than the one currently under consideration. This is the generalized Heegaard splitting obtained from the one under consideration by deleting all components of $\cup_i(\dot{S}_i^j \cup \dot{F}_i^j)$ in C_2 and capping off any components of $\cup_i(\dot{S}_i^j \cup \dot{F}_i^j)$ with annuli. See Figure 10. The upshot is that corresponding to each dipping annulus adjacent to L there is a destabilization of $\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j$. Note that this is also true for any other annular components of $\cup_i S_i - Q$ that meet dipping annuli and are contained in C_2.

Case II: L is not isotopic into ∂M^j.

Denote by $max(L)$ the maximal product neighborhood of L in $M - Q$ that is bounded by annular components of $\cup_i S_i - Q$. See Figures 11 and 12. Then a simpler Heegaard splitting may be constructed by replacing the portion of $\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j$ in $max(L)$ by one corresponding to a simpler schematic. See Figures 13 and 14.

The effect of this "straightening" of $\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j$ in $max(L)$ near a pair of parallel dipping annuli abutting $max(L)$ is pictured in Figures 15 and 16.

The upshot is that with the exception of at most one such pair of dipping annuli, each

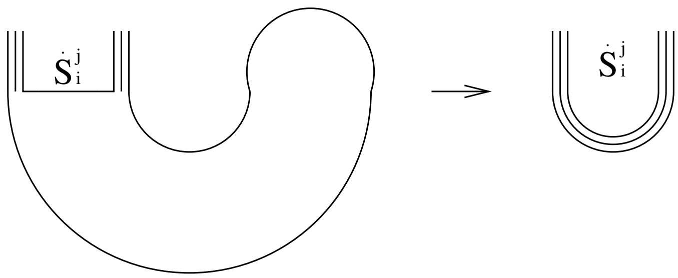

Figure 10: A simpler Heegaard splitting

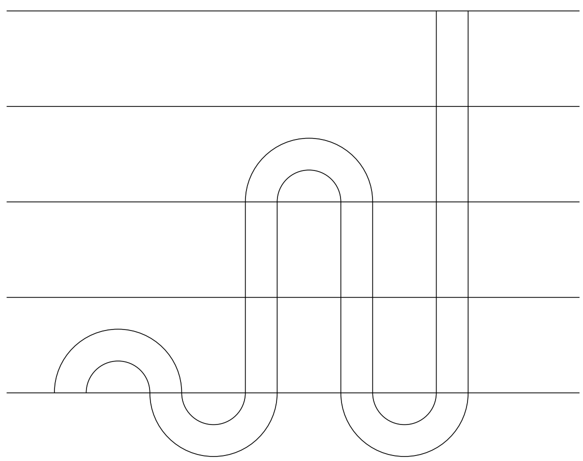

Figure 11: Schematic for $max(L)$

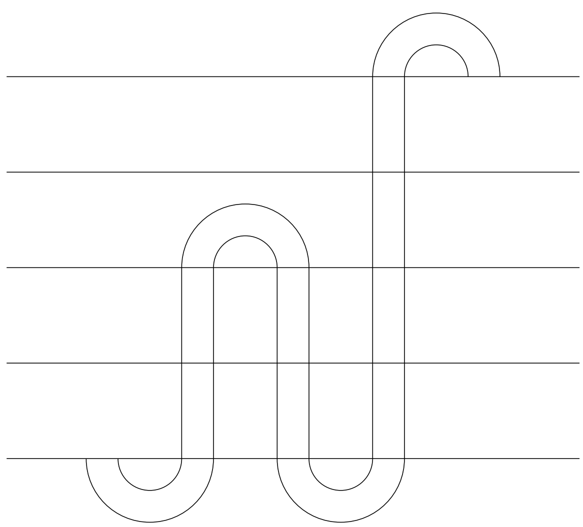

Figure 12: Schematic for $max(L)$

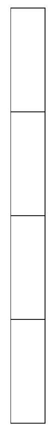

Figure 13: Schematic for a simpler generalized Heegaard splitting near $max(L)$

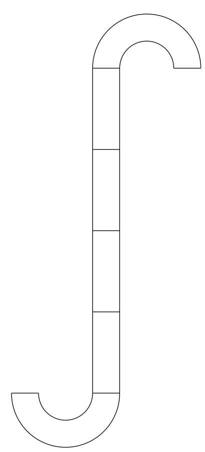

Figure 14: Schematic for a simpler generalized Heegaard splitting near $max(L)$

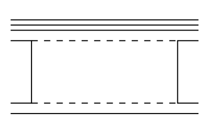

Figure 15: Before the “straightening”

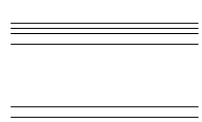

Figure 16: After the “straightening”

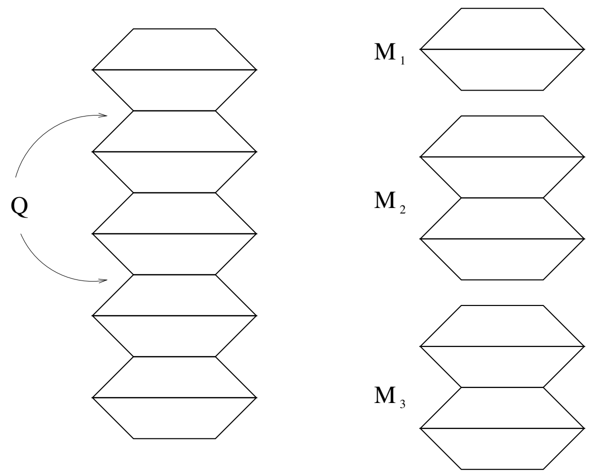

Figure 17: Cutting along a subset of $\mathcal{F}$

annular component of $\cup_i S_i - Q$ abutting dipping annuli corresponds to a destabilization. By Corollary 3.5, this means that we may assume that

$$|\mathcal{A}| \leq \sum_i J(V_i) + 2\sum_j n_j$$

And hence that

$$\sum_i d_i \leq \sum_i (-\chi(Q_{[f_{i-1},s_i]}) + a_i) \leq \sum_i (-\chi(Q_{[f_{i-1},s_i]}) + J(V_i)) + 2\sum_j n_j$$

□

4 Putting it all together

By performing the construction, counting indices, subtracting amounts corresponding to performing destabilizations and genus reductions, we arrive at two propositions that imply the Main Theorem.

Proposition 4.1. *Let M be a compact possibly closed orientable irreducible 3-manifold. Let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ be a strongly irreducible generalized Heegaard splitting. Let Q be a closed essential surface isotopic to a subsurface of $\mathcal{F}$. Then for $M^1, \ldots, M^j$ the completions of the components of $M - Q$ with respect to the path metric there are generalized Heegaard splittings $M^j = (A_1^j \cup_{G_1^j} B_1^j) \cup_{P_1^j} \cdots \cup_{P_{n-1}^j} (A_n^j \cup_{G_n^j} B_n^j)$ for $j = 1, \ldots, k$ such that*

$$\sum_i J(V_i) = \sum_j \sum_i J(A_i^j)$$

Proof: Isotope Q to coincide with a subset of $\mathcal{F}$. Then let M^j be the completion of a component of $M - Q$ and let $V_{i_1}, \ldots, V_{i_l}$, $W_{i_1}, \ldots, W_{i_l}$ be the compression bodies among $V_1, \ldots, V_n, W_1, \ldots, W_n$ that meet M^j. Set $A_i^j = V_i$, if $i \epsilon \{i_1, \ldots, i_l\}$ and $A_i^j = \emptyset$ otherwise, $B_i^j = W_i$, if $i \epsilon \{i_1, \ldots, i_l\}$ and $B_i^j = \emptyset$ otherwise, $G_i^j = S_i$ if $i \epsilon \{i_1, \ldots, i_l\}$ and $G_i^j = \emptyset$ otherwise, and $P_i^j = F_i$ if $i \epsilon \{i_1, \ldots, i_l\}$ and $P_i^j = \emptyset$ otherwise. See Figure 17. □

Proposition 4.2. *Let M be a compact orientable irreducible 3-manifold. Let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ be a strongly irreducible generalized Heegaard splitting. Let Q be a compact boundary incompressible essential surface in M.*

Suppose that $\partial Q \subset \partial_- V$. Suppose further that no component of Q is parallel to a component of $\mathcal{F}$. Denote the completions of the components of $M - Q$ with respect to the path metric by $M^1, \dots, M^k$. Further denote the number of pairwise non isotopic annuli that can be embedded simultaneously in M^j by n_j.

Then there are generalized Heegaard splittings $M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j)$ for M^j for $j = 1, \dots, k$ that satisfy the following inequality:

$$\sum_i J(V_i) \geq \frac{1}{5}\Big(\sum_j \sum_i J(\dot{V}_i^j) + 6\chi(Q) - 8\sum_j n_j\Big)$$

Proof: Since $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is strongly irreducible Q may be isotoped so that each component of $Q \cap (\mathcal{F} \cup \mathcal{S})$ is essential in both Q and $\mathcal{F} \cup \mathcal{S}$ and so that the number of components of $Q \cap (\mathcal{F} \cup \mathcal{S})$ is minimal. By Lemma 2.12, the Main Construction gives generalized Heegaard splittings $M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j)$ for which

$$\sum_j \sum_i J(\dot{V}_i^j) = \sum_i \big(J(V_i) + 2\chi(Q_{[f_{i-1}, s_i]}) + 4(|\alpha_i| + |\beta_i|)\big)$$

By Lemma 3.1 we may choose α_i so that

$$\sum_i |\alpha_i| = \sum_i \big(-\chi(Q_{[f_{i-1}, s_i]}) + d_i\big)$$

and by Lemma 3.2 we may choose $\beta_i = \emptyset$. Thus

$$\sum_i |\beta_i| = 0$$

By Lemma 3.7, we may assume that

$$\sum_i d_i \leq \sum_i \big(-\chi(Q_{[f_{i-1}, s_i]}) + J(V_i)\big) + 2\sum_j n_j$$

Thus

$$\sum_i 4(|\alpha_i| + |\beta_i|) \leq 4\sum_i \big(-\chi(Q_{[f_{i-1}, s_i]}) + d_i\big) \leq$$

$$4\sum_i \big(-2\chi(Q_{[f_{i-1}, s_i]}) + J(V_i)\big) + 8\sum_j n_j$$

Thus

$$\sum_j \sum_i J(\dot{V}_i^j) \leq \sum_i \big(J(V_i) + 2\chi(Q_{[f_{i-1}, s_i]}) - 8\chi(Q_{[f_{i-1}, s_i]}) + 4J(V_i)\big) + 8\sum_j n_j =$$

$$\sum_i \big(5J(V_i) - 6\chi(Q_{[f_{i-1}, s_i]})\big) + 8\sum_j n_j$$

Whence

$$\sum_i J(V_i) \geq \frac{1}{5}\Big(\sum_j \sum_i J(\dot{V}_i^j) + 6\chi(Q) - 8\sum_j n_j\Big)$$

□

Theorem 4.3. *(The Main Theorem) Let M be a compact orientable irreducible 3-manifold. Let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ be a strongly irreducible generalized Heegaard splitting. Let Q be a compact boundary incompressible essential surface in M.*

Suppose that $\partial Q \subset \partial_- V$. Denote the completions of the components of $M - Q$ with respect to the path metric by $M^1, \ldots, M^k$. Further denote the number of pairwise non isotopic annuli that can be embedded simultaneously in M^j by n_j.

Then there are generalized Heegaard splittings $M^j = (A^j_1 \cup_{G^j_1} B^j_1) \cup_{P^j_1} \cdots \cup_{P^j_{n-1}} (A^j_n \cup_{G^j_n} B^j_n)$ for M^j for $j = 1, \ldots, k$ that satisfy the following inequality

$$\sum_i J(V_i) \geq \frac{1}{5}\Big(\sum_j \sum_i J(\dot{A}^j_i) + 6\chi(Q) - 8\sum_j n_j\Big)$$

Proof: By Lemma 1.11, Q may be isotoped so that each component of $Q \cap (\mathcal{F} \cup \mathcal{S})$ is essential in both Q and $\mathcal{F} \cup \mathcal{S}$. We may assume that the number of components of $Q \cap (\mathcal{F} \cup \mathcal{S})$ is minimal subject to this condition.

Partition Q into $Q^p \sqcup Q^n$, where Q^p consists of those components of Q that are parallel to a component of $\mathcal{F}$ and Q^n consists of those components of Q that are not parallel to any component of $\mathcal{F}$. Then proceed first as in Proposition 4.1 using Q^p instead of all of Q. This yields an equality. In each of the resulting 3-manifolds proceed as in Proposition 4.2 using the appropriate subset of Q^n instead of all of Q. This yields the required inequality. $\square$

Theorem 4.4. *Let M be a compact orientable irreducible 3-manifold. Let Q be a boundary incompressible essential surface in M. Denote the completions of the components of $M - Q$ with respect to the path metric by $M^1, \ldots, M^k$. Further denote the number of pairwise non isotopic annuli that can be embedded simultaneously in M^j by n_j.*

Then

$$g(M, \partial M) \geq \frac{1}{5}\Big(\sum_j g(M^j, \partial M^j) - |M - Q| + 5 - 2\chi(\partial_- V) + 4\chi(Q) - 4\sum_j n_j\Big)$$

Proof: Let $M = V \cup_S W$ be a Heegaard splitting that realizes $g(M, \partial M)$. Let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ be a weak reduction of $M = V \cup_S W$. Then

$$\sum_{i=1}^{m} J(V_i) = 2g - 2 + \chi(\partial_- V)$$

Now apply Theorem 4.3 to $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$. This yields generalized Heegaard splittings $M^j = (A^j_1 \cup_{G^j_1} B^j_1) \cup_{P^j_1} \cdots \cup_{P^j_{n-1}} (A^j_n \cup_{G^j_n} B^j_n)$ for which

$$\sum_i J(V_i) \geq \frac{1}{5}\Big(\sum_j \sum_i J(\dot{A}^j_i) + 6\chi(Q) - 8\sum_j n_j\Big)$$

Amalgamating $M^j = (A^j_1 \cup_{G^j_1} B^j_1) \cup_{P^j_1} \cdots \cup_{P^j_{n-1}} (A^j_n \cup_{G^j_n} B^j_n)$ yields Heegaard splittings $M^j = V^j \cup_{S^j} W^j$ with

$$\sum_i J(A^j_i) = 2g(S^j) - 2 + \chi(\partial_- V^j)$$

Hence

$$2g(S) - 2 + \chi(\partial_- V) = \sum_i J(V_i) \geq$$

$$\frac{1}{5}(\sum_j \sum_i J(\dot{A}_i^j) + 6\chi(Q) - 8\sum_j n_j) =$$

$$\frac{1}{5}(\sum_j (2g(S^j) - 2 + \chi(\partial_- V^j)) + 6\chi(Q) - 8\sum_j n_j)$$

Here

$$\sum_j \chi(\partial_- V^j) = \chi(\partial_- V) + 2\chi(Q)$$

Therefore

$$2g(S) - 2 + \chi(\partial_- V) \geq \frac{1}{5}(\sum_j (2g(S^j) - 2) + \chi(\partial_- V) + 8\chi(Q) - 8\sum_j n_j)$$

Hence

$$2g(S) \geq \frac{1}{5}(\sum_j (2g(S^j) - 2) + 10 - 4\chi(\partial_- V) + 8\chi(Q) - 8\sum_j n_j)$$

Whence

$$g(M, \partial M) \geq \frac{1}{5}(\sum_j g(M^j, \partial M^j) - |M - Q| + 5 - 2\chi(\partial_- V) + 4\chi(Q) - 4\sum_j n_j)$$

□

Finally, we consider two interesting cases encompassed by this construction. In [4], M. Eudave-Muñoz constructs a family of tunnel number 1 knots whose complements contain incompressible surfaces of arbitrarily high genus. For elementary definitions pertaining to knot theory, see for instance [2] or [8].

Remark 4.5. *Let K be a tunnel number one knot whose complement $C(K) = S^3 - \eta(K)$ contains a closed incompressible surface F of genus g. Let M^1, M^2 be the manifolds obtained by cutting $C(K)$ along F. (Here M^1, say, is the complement of an open regular neighborhood of the toroidal graph in the construction of Eudave-Muñoz and M^2 is the component containing $\partial C(K)$.) In the examples of Eudave-Muñoz the genus is $g(F) + 1$ for both M^1 and M^2, i.e.,*

$$g(M^1) + g(M^2) = 2g(F) + 2$$

This number is significantly lower than the upper bound derived here. This leads one to believe that the bound suggested is too high. However, the constructions in this paper show that the upper bound in the inequality would be obtained if the splitting surface of a minimal genus Heegaard splitting had to intersect the incompressible surface many times. The examples of Eudave-Muñoz are very well behaved in this respect, as the splitting surface of the genus 2 Heegaard splitting may be isotoped to intersect the incompressible surface in only one curve inessential in F or two curves essential in F.

Remark 4.6. *Consider $R \times I$ for R a closed connected orientable surface. The genus of $R \times I$ is $g(R)$. On the other hand, the genus of the manifold M obtained by identifying $R \times \{0\}$ to $R \times \{1\}$ is 2 for a suitably chosen gluing homeomorphism. For details on choosing such a gluing homeomorphism, see [17]. Here the right hand side in Theorem 4.4 is negative unless R is a torus. This is disconcerting, but merely means that the result is automatically true. Again, the genus 2 Heegaard splitting in this example is well behaved with respect to its intersection with R; it too may be isotoped so that it intersects R either in one curve inessential in R or two curves essential in R.*

Recall that in the appendix to [14], A. Casson provided a class of 3-manifolds each having a strongly irreducible generalized Heegaard splitting and each containing an essential annulus such that the number of dipping annuli can be arbitrarily large. These examples provide one reason to believe the upper bound provided here is not larger than necessary.

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