# Waldhausen's "Heegaard-Zerlegungen der 3-Sphäre"

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**Abstract**

We provide an annotated and illustrated translation of key passages of Friedhelm Waldhausen's paper that establishes the uniqueness of Heegaard splittings of the 3-sphere.

## 1 Introduction

In 1968 Friedhelm Waldhausen published a paper entitled "Heegaard-Zerlegungen der 3-Sphäre", *i.e.*, "Heegaard splittings of the 3-sphere". This famous paper establishes the uniqueness of Heegaard splittings of a given genus for the 3-sphere. In other words, there is only one Heegaard splitting of the 3-sphere of any given genus $g$ and it is the $g$-fold stabilization of the genus 0 splitting.

Other proofs of this theorem have been given since then, most notably by Scharlemann and Thompson in [4]. Yet Waldhausen's original proof remains of foremost interest. His strategy may generalize to prove similar results in situations where Scharlemann and Thompson's strategy fails to apply. Waldhausen's strategy entails applying the Reidemeister-Singer Theorem to compare a given Heegaard splitting of $S^3$ to the genus 0 Heegaard splitting. Two collections of stabilizing pairs of disks are then compared and played off against each other.

Recent years have seen a resurgence of interest in issues pertaining to stabilization. So an English version of Waldhausen's paper is overdue. Some of Waldhausen's terminology is outdated. We will provide the definitions needed according to current custom and usage. The aim here is not a literal translation of Waldhausen's entire paper. The section on definitions follows the spirit of Waldhausen's section on definitions, but it has been updated to conform to current usage. Our sections on equivalence and stable equivalence, Heegaard splittings of the 3-sphere and on remarks provide a literal translation of the bulk of the corresponding sections of Waldhausen's paper. Passages from Waldhausen's paper appear in quotations. In accordance with current custom, we have added figures to illustrate examples of the constructions described. Waldhausen's paper, in accordance with the custom of the time, contained no figures at all. We have ignored some passages, especially those observing facts now considered standard, *e.g.*, those on general position and the like, entirely. Waldhausen's paper has stood the test of time. It is as relevant today as it was when it was published. We hope that this updated and illustrated translation will remind the reader of the insights of Waldhausen's paper and prove to be a valuable reference.

# 2 Definitions

A *handlebody* is a 3-manifold that is homeomorphic to a regular neighborhood of a connected graph in the 3-sphere. A Heegaard splitting of a 3-manifold $M$ is a pair $M = V \cup_F W$, such that 1) $V, W$ are handlebodies, 2) $M = V \cup W$, 3) $V \cap W = \partial V = \partial W$ is a surface that we denote by $F$. Here $F$ is called the *splitting surface*. Two Heegaard splittings are considered *equivalent* if their splitting surfaces are isotopic. We will occasionally be interested in oriented equivalence. Two Heegaard splittings $M = V \cup_F W$ and $M = V' \cup_{F'} W'$ are considered *orientedly equivalent* if the isotopy takes $V$ to $V'$ and $W$ to $W'$.

A Heegaard splitting $M = V \cup_F W$ is *stabilized* if there is a pair of disks $(D, E)$ with $D \subset V$ and $E \subset W$ such that $\#|\partial D \cap \partial E| = 1$. We call the pair of disks $(D, E)$ a stabilizing pair of disks. A Heegaard splitting that is not stabilized is *unstabilized.*

**Remark 1.** *Waldhausen uses the antiquated expression "minimal Heegaard splitting" for a Heegaard splitting that is not stabilized. He is quick to point out that according to his terminology, two "minimal Heegaard splittings" need not have the same genus. It is precisely this fact, that a "minimal Heegaard splitting" need not have minimal genus, that forced this usage into oblivion. Current usage defines a minimal Heegaard splitting to be a Heegaard splitting of minimal genus. The relevant terms here, according to current usage, are "stabilized" and "unstabilized".*

Let $S^3 = X \cup_T Y$ be the Heegaard splitting of $S^3$ such that $X$ (and hence also $Y$) is an unknotted solid torus. Note that this is the only Heegaard splitting of $S^3$ of genus 1. There is a stabilizing pair of disks for this Heegaard splitting. The complement of this stabilizing pair of disks is a 3-ball that meets $T$ in a disk. Now given a Heegaard splitting $M = V \cup_F W$, the pairwise connect sum $(M, F)\#(S^3, T)$ defines a Heegaard splitting. The stabilizing pair of disks survives in the new Heegaard splitting. Conversely, if a Heegaard splitting $M = V' \cup_{F'} W'$ is such that $(M, F') \sim (M, F)\#(S^3, T)$ for some Heegaard splitting $M = V \cup_F W$, then $M = V' \cup_{F'} W'$ is stabilized. Waldhausen emphasizes the fact that if $(M, F') \sim (M, F)\#(S^3, T)$ for some Heegaard splitting $M = V \cup_F W$, it does not follow that $M = V \cup_F W$ unique.

**Remark 2.** *Waldhausen defines stabilization in terms of this connected sum.*

"We say that $M = V \cup_F W$ is *stabilized* if there is a Heegaard splitting $(M, F')$ such that $(M, F) \sim (M, F')\#(S^3, T)$. Recursively, we define

$$(M, F)\#n(S^3, T) \sim ((M, F)\#(n-1)(S^3, T))\#(S^3, T)$$

We call two Heegaard splittings $M = V \cup_F W$ and $M = V' \cup_{F'} W'$ *stably equivalent* if there are $m, n$ such that

$$(M, F)\#n(S^3, T) \sim (M, F')\#m(S^3, T)).$$

Recall the Reidemeister-Singer Theorem: If $M = V \cup_F W$ and $M = V' \cup_{F'} W'$ are Heegaard splittings of $M$, then $M = V \cup_F W$ and $M = V' \cup_{F'} W'$ are stably equivalent. See [3], [5], cf page 56 of [7]. (Note that Singer employs homeomorphism, rather than isotopy, for his equivalence classes. But the above mentioned theorem is nevertheless proven in [5] on pages 107-111.)"

# 3 Equivalence and stable equivalence

" ... the statement $(M, F'') \sim (M, F')\#n(S^3, T)$ is equivalent to the statement that there are $n$ stabilizing pairs of disks in $(M, F'')$. - But a "system of $n$ stabilizing pairs of disks" is a rather cumbersome term. Providing and characterizing a more streamlined definition occupies the rest of this section.

(2.1) Let $M = V \cup_F W$ be a Heegaard splitting ... A system of $n$ disjoint disks $v = v_1 \cup \cdots \cup v_n$ in $V$ is called a *good system of $n$ meridian disks in $V$* if there is a system of $n$ disjoint disks $w = w_1 \cup \cdots \cup w_n$ in $W$ such that (after renumbering the components of $v$ and $w$, if necessary):

$\partial v_j \cap \partial w_j$ consists of exactly one point

$\partial v_i \cap \partial w_j = \emptyset$ when $i > j$;

here $w$ is called a *system corresponding to $v$*.

If $v$ is a system of $n$ meridian disks in $V$ and $w$ is a system corresponding to $v$, then $w$ is necessarily a good system of $n$ meridian disks in $W$ and $v$ is a system corresponding to $w$."

**Remark 3.** *Note that $\partial v_i \cap \partial w_j = \emptyset$ is only required for $i > j$. The fact that this assumption implies the existence of a good system of $n$ meridian disks such that $\partial v_i \cap \partial w_j = \emptyset$ when $i \neq j$ is the subject of the following lemma. The gist of the proof a disk slide. It is described explicitly.*

"(2.2) Lemma. Let $v$ be a good system of $n$ meridian disks in $V$ and let $w$ be a system corresponding to $v$.

(1) There is a system $\tilde{w}$ corresponding to $v$ such that $\partial v_j \cap \partial \tilde{w}_j$ is exactly one point and such that $\partial v_i \cap \partial \tilde{w}_j = \emptyset$ for $i \neq j$.

(2) For $\tilde{U} = U(F \cup v \cup w)$ a regular neighborhood of $F \cup v \cup w$ in $M$ and $\tilde{F} = \partial U \cap V$, $\tilde{U}$ is homeomorphic to $\tilde{F} \times I$. (Where $I$ is the unit interval.)

Proof of (1): We will construct the system $\tilde{w}$ from the system $w$.

Let $i$ be the smallest index such that there is a point of intersection $q \subset \partial v_i \cap \partial w_j$, $j > i$. Let $k$ be one of the two arcs in $\partial v_i$ that connects $q$ with the point of intersection of $\partial v_i$ with $\partial w_i$. Suppose $q$ and $k$ have been chosen so that $w$ misses the interior of $k$. Let $U(w_i \cup k \cup w_j)$ be a regular neighborhood of $w_i \cup k \cup w_j$ in $M$. Then $W \cap U(w_i \cup k \cup w_j)$ consists of three disks. Of these the first is isotopic to $w_i$, the second is isotopic to $w_j$; denote the third by $w_j'$. For each $h \neq i$, $\partial v_h \cap \partial w_j'$ consists of the same number of points as $\partial v_h \cap \partial w_j$; but $\partial v_i \cap \partial w_j'$ contains exactly one point less than $\partial v_i \cap \partial w_j$. We replace $w_j$ by $w_j'$. We repeat this procedure as often as possible. The result is a system $\tilde{w}$ with the stated properties."

**Remark 4.** *See Figures 1 and 2.*

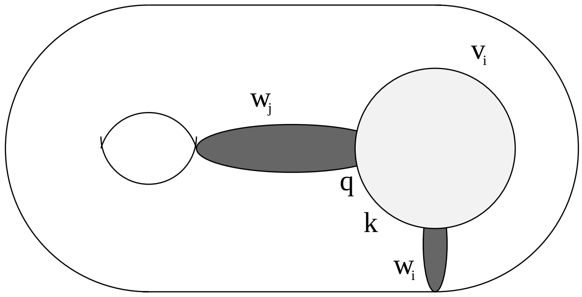


Figure 1: *The disks* $v_i, w_j$ *and* $w_i$

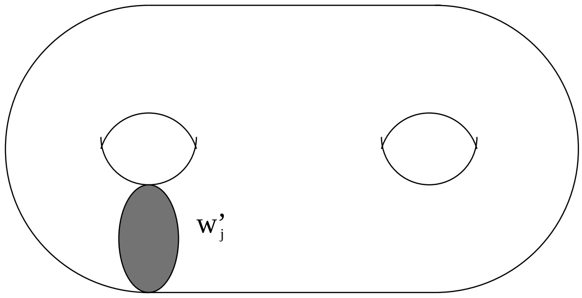


Figure 2: *The disk* $w'_j$

"Proof of (2): a) We first check that the change in $w$ described in (1) leaves the homeomorphism type of $U(F \cup v \cup w)$ unchanged.

Let $U^*$ be a small regular neighborhood of $F \cup v \cup w_1 \cup \cdots \cup w_i$. ... Let $W^* = W - interior(U^*)$. Then $U(F \cup v \cup w)$ is homeomorphic to the union of $U^*$ and a regular neighborhood of

$$X^* = (U^* \cap W^*) \cup (w_{i+1} \cap W^*) \cup \cdots \cup (w_n \cap W^*)$$

in $W^*$. But $X^*$ is isotopic to

$$(U^* \cap W^*) \cup (w_{i+1} \cap W^*) \cup \cdots \cup (w'_j \cap W^*) \cup \cdots \cup (w_n \cap W^*)$$

in $W^*$.

b) It follows that we may replace $w$ with $\tilde{w}$ in our proof of the above assertion. Let $U_j$ be a regular neighborhood of $v_j \cap \tilde{w}_j$ (in M). Then $U_j$ is a 3-ball; $\partial U_j \cap V$ and $\partial U_j \cap W$ are disks; $U_j \cap F$ is a once punctured torus. Since the $U_j$'s are pairwise disjoint, and since $U(F \cup v \cup \tilde{w})$ is homeomorphic to a regular neighborhood of $F \cup (\cup_j U_j)$, the assertion follows."

**Remark 5.** *Here Waldhausen is using the connected sum definition of stabilization. He is exhibiting* $(M, F)$ *as* $(M, \tilde{F}) \# n(S^3, T)$.

"(2.3) Let $M = V \cup_F W$ be a Heegaard splitting ... Let $v$ be a good system of $n$ meridian disks in $V$ and let $w$ be a system corresponding to $v$; let $U(v)$ and $U(w)$ be regular neighborhoods of $v$ and $w$ in $M$. Set

$$\tilde{V} = V - int(U(v)),\ \tilde{W} = closure(M - \tilde{V})$$

$$W^* = W - int(U(w)),\ V^* = closure(M - W^*)$$

Since $v$ does not separate $V$, $\tilde{V}$ is a handlebody [[8], Corollary 1]; the same is true for $W^*$. On the other hand, in accordance with (2.2), the submanifolds $\tilde{V}$ and $V^*$ are isotopic in $M$. Thus if $\tilde{F}$ is the surface $\partial\tilde{V}$ together with an orientation, then $M = \tilde{V} \cup_{\tilde{F}} \tilde{W}$ is a Heegaard splitting. If $M = \tilde{V} \cup_{\tilde{F}} \tilde{W}$ is orientedly equivalent to $M = \tilde{W} \cup_{\tilde{F}} \tilde{V}$ then the equivalence class of $M = \tilde{V} \cup_{\tilde{F}} \tilde{W}$ is determined by $M = V \cup_F W$ and $v$ or $M = V \cup_F W$ and $w$. If $M = \tilde{V} \cup_{\tilde{F}} \tilde{W}$ is not orientedly equivalent to $M = \tilde{W} \cup_{\tilde{F}} \tilde{V}$ then we distinguish one of the orientations via the following rule: If the orientations of $M$ and $F$ determine $V$ as the first of the two handlebodies $V, W$, then the orientations of $M$ and $\tilde{F}$ determine $\tilde{V}$ as the first of the two handlebodies $\tilde{V}, \tilde{W}$.

In either case, either $M = V \cup_F W$ and $v$ or $M = V \cup_F W$ and $w$ determine a Heegaard splitting $\tilde{V} \cup_{\tilde{F}} \tilde{W}$ up to isotopy; we refer to this Heegaard splitting as arising from $M = V \cup_F W$ by a *reduction along* $v$ and write

$$(M, \tilde{F}) \sim (M, F(v)) \text{ or } (M, \tilde{F}) \sim (M, F(w)),\ \textit{respectively}.$$

(Caution: It is true that $(M, (-F)(v)) \sim (M, -(F(v)))$; but $(M, F) \sim (M, -F)$ does not necessarily imply that $(M, F(v)) \sim (M, -F(v))$.)

The above discussion implies the following:

(2.4) Lemma. Let $M = V \cup_F W$ be a Heegaard splitting ... Let $v$ and $v'$ be good systems of meridian disks in $V$ (where possibly $v \cap v' \neq \emptyset$). Suppose there is a system $w$ in $W$ that corresponds to both $v$ and $v'$. Then $(M, F(v)) \sim (M, F(v'))$.

As a corollary of our definitions we obtain the first part of the following lemma:

(2.5) Lemma. Let $(M, F_1)$ and $(M, F_2)$ be stably equivalent Heegaard splittings.

Then there is a Heegaard splitting $(M, F)$ and good systems $v$ and $x$ of meridian disks in $V$, a system $w$ corresponding to $v$, a system $y$ corresponding to $x$, (where $V \cup W = M$, $V \cap W = F$), such that

(1) $(M, F(v)) \sim (M, F_1)$ and $(M, F(x)) \sim (M, F_2)$,

(2) $v \cap x = \emptyset$ and $w \cap y = \emptyset$.

Proof of (2): Let $(M, F), v, w, x, y$ be as in (1). We show how to alter these so as to satisfy (2). ...

a) Suppose there are closed curves in $v \cap x$. Then there is a disk $D$ in $int(v)$ such that $D \cap x = \partial D$. Here $\partial D$ bounds a disk $D'$ in $x$. We construct $x'$ from $x$ by replacing $D'$ by $D$ and isotoping $D$ off of $v$. By (2.4),

$$(M, F(x')) \sim (M, F(x)).''$$

**Remark 6.** *This is an example of what is called a "standard innermost disk argument".*

"b) Suppose there is an arc $k$ in $v \cap x$. Suppose the components $v_1, \ldots, v_n$ of $v$ and $w_1, \ldots, w_n$ of $w$ are numbered so that $\partial v_h \cap \partial w_i$ consists of exactly one point of intersection when $h = i$ and is empty when $h > i$. Suppose that $k$ lies in $v_j$. Let $U(k)$ be a regular neighborhood of $k$ in $M$. Let $\bar{w}$ be a disk in $U(k) \cap int(V)$ such that

$\bar{w} \cap \partial U(k) = \partial \bar{w}$ in $\partial U(k) \cap int(V)$ is not simply connected and meets $v_j$ in exactly two points."

**Remark 7.** *See Figure 3.*

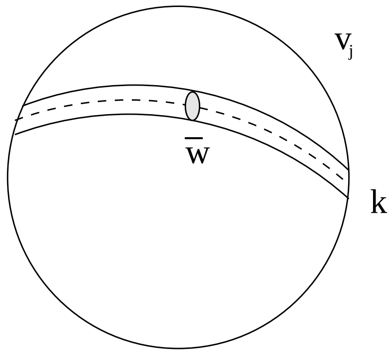


Figure 3: *The disk $v_j$ (containing the arc $k$) and the disk $\bar{w}$*

"Clearly $W' = W \cup U(k)$ is a handlebody; the same holds for $V' = V - U(k)$, since the arc $k$ lay in the disk $v_j$. We set $F' = V' \cap W'$ and orient $F'$ via the induced orientation on $F' \cap F$. We further set

$$v'_i = v_i \text{ and } w'_i = w_i, \text{ for } i < j$$

$$v'_{i+1} = v_i \text{ and } w'_{i+1} = w_i, \text{ for } i > j.$$

Here $v_j \cap V'$ consists of two disks. We denote the one containing the point of intersection with $w_j$ by $v'_j$ and the other by $v'_{j+1}$. Finally, we set

$$w'_j = w_j \text{ and } w'_{j+1} = \bar{w}.$$

Now $v'$ is a good system of $(n+1)$ disks in $V'$ and $w'$ is a system corresponding to $v'$; clearly

$$(M, F'(v')) \sim (M, F(v)).$$

Analogous to constructing $v'$ and $w'$ from $v$ and $w$ we construct $x'$ and $y'$ from $x$ and $y$; (in the construction of $y'$ a disk analogous to $\bar{w}$ is chosen, but we choose this disk to be disjoint from the actual $\bar{w}$).

In each step in (a) and (b) the number of components of $(v \cap x) \cup (w \cap y)$ is lowered by at least one; hence after a finite number of steps $v \cap x = \emptyset$. In the same manner we obtain $w \cap y = \emptyset$."

# 4 Heegaard splittings of the 3-sphere

"(3.1) Theorem. Let $M = X \cup_G Y$ be an unstabilized Heegaard splitting of the 3-sphere. Then $G$ has genus 0.

It seems that noone has ever doubted that this theorem holds. For some time it was assumed to be obvious (see [[3], footnote on page 193]). It was Reidemeister who first indicated that there is in fact something to prove, [[3], page 192]. Lately, references have become more cautious, [[2], (16.4)], [[6], Conjecture A].

In [[3], page 192] a proof is outlined. But it is questionable whether the indicated method (studying certain level sets) can be turned into a proof. Arguing in this fashion does not exploit the fact that the underlying manifold is $S^3$. One merely compares an unknown Heegaard splitting to a Heegaard splitting of genus $\leq 1$; hence there is a danger that one is arguing against the existence of the example mentioned in (4.4.1) below.

Proof of (3.1): Let $M = X' \cup_{G'} Y'$ be the Heegaard splitting of genus 0. By the Reidemeister-Singer Theorem $M = X \cup_G Y$ and $M = X' \cup_{G'} Y'$ are stably equivalent. Hence by (2.5) there is a Heegaard splitting $M = V \cup_F W$ with the following properties:

For ... $n$ the genus of $F$, there is a good system $v = v_1 \cup \cdots \cup v_n$ in $V$ and a system $w = w_1 \cup \cdots \cup w_n$ corresponding to $v$ and a good system $x = x_1 \cup \cdots \cup x_n$ in $V$ and a system $y = y_1 \cup \cdots \cup y_n$ corresponding to $x$ such that $(M, G) \sim (M, F(x))$ and such that $v \cap x = \emptyset$ and $w \cap y = \emptyset$.

We will assume that among all Heegaard splittings with the above properties, $M = V \cup_F W$ has been chosen to minimize $n$. We further assume that $n > 0$. We will show that these assumptions lead to a contradiction.

... Suppose the components of $v, w, x, y$ are numbered so that $\partial v_i \cap \partial w_j$, respectively $\partial x_i \cap \partial y_j$, consists of exactly one point of intersection in the case that $i = j$ and is empty for $i > j$.

(3.2) Through a modification of $y$ alone we ensure that $y \cap v_n$ consists of at most one point, (while maintaining $y \cap w = \emptyset$). Proof by induction on the number of points in $y \cap v_n$:

Case 1: *The component $y_j$ of $y$ meets $v_n$ in at least two points.* Since $v_n \cap w$ is a single point there is an arc $k$ in $\partial v_n$ that is disjoint from $w$ such that $k \cap y_j = \partial k$."

**Remark 8.** *See Figure 4.*

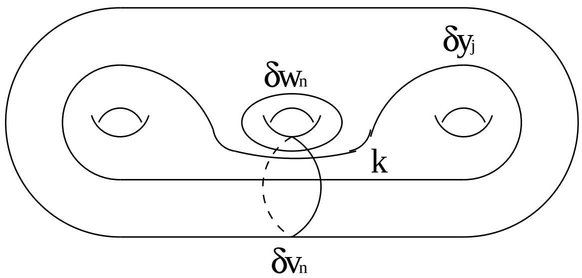


Figure 4: *The arc k*

"Let $U(w)$ be a regular neighborhood of $w$ in $M$; then $y_j \cup k$ is contained in $W - U(w)$. Here $W - interior(U(w))$ is a 3-ball that $y_j$ cuts into two 3-balls, (this is where we use the fact that $M$ is the 3-sphere). Hence there is a disk $D$ in $W - U(w)$, such that $D \cap \partial W = k$ and $D \cap y_j = \partial D \cap y_j = closure(\partial D - k)$."

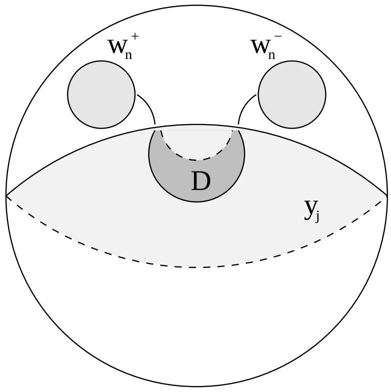


Figure 5: *The arc $D$*

**Remark 9.** *See Figure 5.*

"After a deformation of $D$ that fixes $\partial D$ (and after removing any closed curves of intersection that may arise in the standard manner) $D \cap y$ is a system of disjoint simple arcs none of which ends on $D \cap y_j$; hence we may assume (possibly after replacing $y_j$ with some other component of $y$) that $D \cap y_j = D \cap y$.

Let $U(D \cup y_j)$ be a regular neighborhood of $D \cup y_j$ in $M$. Then $W \cap \partial U(D \cup y_j)$ consists of three disks. One of these intersects $v_n$ in the same number of points as $y_j$ and is isotopic to $y_j$ in $W$. Let $y_j^1$ and $y_j^0$ be the other two. Then $y_j^1 \cup y_j^0$ and $v_n$ intersect in two fewer points than $y_j$ and $v_n$. For each $i$, $(y_j^1 \cup y_j^0) \cap x_i$ consists of the same number of points as $y_j \cap x_i$. In particular, $(y_j^1 \cup y_j^0) \cap x_j$ consists of exactly one point; we may assume this point lies in $y_j^1$. We replace $y_j$ with $y_j^1$.

Case 2:*Every component of $y$ meets $v_n$ in at most one point; but $y \cap v_n$ consists of at least two points.* Then there is an arc $k$ in $\partial v_n$ disjoint from $w$ such that

$$\partial k = k \cap y = (k \cap y_i) \cup (k \cap y_j);$$

where we may assume that $i < j$."

**Remark 10.** *See Figure 6.*

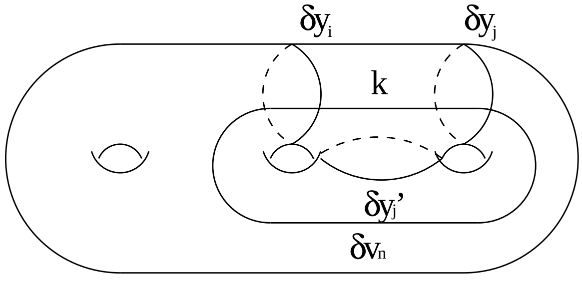


Figure 6: *The arc $k$ along with the boundaries of $v_n, y_i, y_j, y_j'$*

"Suppose that $U(y_i \cup k \cup y_j)$ is a regular neighborhood of $y_i \cup k \cup y_j$ in $M$. Then $W \cap \partial U(y_i \cap k \cup y_j)$ consists of 3 disks. The first of these is isotopic to $y_i$, the second to $y_j$; denote the third by $y_j'$. Then $y_j' \cap v_n = \emptyset$. The points of intersection of $y_j'$ with

$x_h$ correspond to those of $y_i \cup y_j$ with $x_h$. Hence $y$ remains a system corresponding to $x$ when we replace $y_j$ with $y'_j$.

(3.3) Case 1: $y \cap v_n \neq \emptyset$, hence by (3.2) it is exactly one point; we assume that this point lies in $y_j$.

a) We replace $x$ and $y$ by $x'$ and $y'$ as follows:

$$x'_m = v_n \text{ and } y'_m = y_j,$$
$$x'_i = x_i \text{ and } y'_i = y_i, \text{ if } i < j,$$
$$x'_{i-1} = x_i \text{ and } y'_{i-1} = y_i, \text{ if } j < i \leq m.$$

$x'$ is a good system of meridian disks and $y'$ is a system corresponding to $x'$; by (2.4)

$$(M, F(x')) \sim (M, F(x)) \sim (M, G).$$

b) We retain $x'$ and replace $y'$ by $y''$ by setting

$$y''_m = w_n$$

and $y''_i = y'_i$ for $i < m$; $y''$ is also a system corresponding to $x'$.

c) Let $U(v_n \cup w_n)$ be a regular neighborhood of $v_n \cup w_n = x'_m \cup y''_m$ in $M$. Let $\tilde{V} = V - int(U(v_n \cup w_n))$ and $\tilde{W} = W \cup U(v_n \cup w_n)$; let $\tilde{F} = \tilde{V} \cap \tilde{W}$, with the orientation induced by $\tilde{F} \cap F$.

Since $v_n \cap w = v_n \cap w_n$ and $v_n \cap y'' = v_n \cap y'_m$, we have

$$w_i \cap \partial\tilde{W} = \partial w_i \text{ for } i < n \text{ and}$$
$$y''_i \cap \partial\tilde{W} = \partial y''_i \text{ for } i < m.$$

Thus $\tilde{v} = \tilde{v}_1, \ldots, \tilde{v}_{n-1} = v'_1 \cap \tilde{V}, \ldots, v_{j-1} \cap \tilde{V}$ is a good system of $n-1$ meridian disks in $\tilde{V}$ and $\tilde{w} = w_1, \ldots, w_{n-1}$ is a system corresponding to $\tilde{v}$.

The same holds for

$$\tilde{x} = \tilde{x}_1, \ldots, \tilde{x}_{m-1} = x'_1 \cap \tilde{V}, \ldots, x'_{m-1} \cap \tilde{V} \text{ and } \tilde{y} = y''_1, \ldots, y''_{m-1}.$$

Here $(M, \tilde{F}(\tilde{x})) \sim (M, F(x))$; furthermore, $\tilde{v} \cap \tilde{x} = \emptyset$ and $\tilde{w} \cap \tilde{y} = \emptyset$. Since $\tilde{F}$ has smaller genus than $F$, this is a contradiction to our choice of $M = V \cup_F W$.

Case 2: $y \cap v_n = \emptyset$. We define

$$x^*_{m+1} = v_n \text{ and } y^*_{m+1} = w_m$$
$$x^*_i = x_i \text{ and } y^*_i = y_i \text{ for } i \leq m.$$

Here $x^*$ is a good system of $m+1$ meridian disks in $V$ and $y^*$ is a system corresponding to $x^*$.

If follows that $(M, G) \sim (M, F(x)) \sim (M, F(x^*))\#(S^3, T)$, contradicting the assumption that $(M, G)$ is minimal."

# 5 Remarks

"(4.1) Haken [[1], Section 7] has shown the following: Let $M = V \cup_F W$ be a Heegaard splitting; suppose there is a 2-sphere in $M$ that does not bound a 3-ball. Then there is such a 2-sphere that meets $F$ in a single curve. - From this and (3.1) is follows that the manifolds $S^1 \times S^2 \# \ldots \# S^1 \times S^2$ possess only the known Heegaard splittings.

(4.2) Let $N$ be a handlebody and let $D$ be a system of disks in $N$ such that $D \cap \partial N = \partial D$. A manifold that is homeomorphic to a regular neighborhood of $D \cup \partial N$ in $N$ is called a *compression body* and the boundary component corresponding to $\partial N$ is called its *distinguished boundary.*

Let $M$ be an oriented manifold and let $F$ be an oriented closed surface in $M$ that cuts $M$ into two components, $V$ and $W$, such that each of $V$ and $W$ is either a handlebody or a compression body with $F$ as its distinguished boundary; suppose that the orientations specify $V$ as the "first" part. Then we call $M = V \cup_F W$ a *Heegaard splitting relative to the partition* $(V \cap \partial M, W \cap \partial M)$ *of* $\partial M$. We define the notions of "equivalence", "$\#(S^3, T)$" and "stable equivalence" as for closed manifolds.

The Reidemeister-Singer Theorem holds in the form: Let $M = V \cup_F W$ and $M = V' \cup_{F'} W'$ be Heegaard splittings of $M$ relative to the partition $(G_1, G_2)$ of $\partial M$, then $M = V \cup_F W$ and $M = V' \cup_{F'} W'$ are stably equivalent. Section 2 applies almost verbatim (though all disks must of course be required to lie in the interior of the manifold).

(4.3) The result of Haken mentioned in (4.1) also holds for Heegaard splittings of manifolds with boundary (though in the statement "only" must be replaced with "at most"); the proof is similar. In the same fashion one establishes the following: Let $M = V \cup_F W$ be a Heegaard splitting; suppose there is a disk $D$ such that $D \cap \partial M = \partial D$ does not bound a disk on $\partial M$. Then there is such a disk that meets $F$ in only one curve. - From this and (3.1) we obtain the following: If $M = V \cup_F W$ is an unstabilized Heegaard splitting of a handlebody then $F$ is parallel to $\partial M$.

(4.4) We define a different notion of equivalence than the one used so far, namely orientation preserving homeomorphism of pairs; for manifolds with boundary the partition of the boundary must also be preserved. (How or to what degree the two notions of equivalence coincide is entirely unknown.) The connect sum then turns equivalence classes of Heegaard splittings into elements of a (commutative and associative) monoid.

(1) The cancellation law is invalid in this monoid. For example, let $M = V \cup_F W$ be a Heegaard splitting of genus 1 of a lens space not equal to $S^3$. Then $M = V \cup_F W$ is (in accordance with our new notion of equivalence) characterized by a pair of relatively prime numbers $(\alpha, \beta)$, $0 < \beta < \alpha$. Since $M = W \cup_F V$ is characterized by $(\alpha, \beta')$ where $\beta\beta' \equiv 1 \; mod \; \alpha$, $M = V \cup_F W$ and $M = W \cup_F V$ are generically not equivalent. On the other hand, $(M, F)\#(S^3, T) \sim (M, -F)\#(S^3, T)$. For suppose $D$ is a disk in $F$ and let $U(F - int(D))$ be a regular neighborhood of $F - int(D)$ in $M$, and set $G = \partial U(F - int(D))$. With an appropriate choice of orientation of $G$, $(M, G)$ is a representative of both $(M, F)\#(S^3, T)$ and $(M, -F)\#(S^3, T)$.

(2) Let $M = V \cup_F W$ and $M' = V' \cup_{F'} W'$ be Heegaard splittings of genus 1 of the lens spaces $(5, 2)$ and $(7, 2)$. By taking connect sums we obtain four Heegaard splittings of the oriented manifold $M \# M'$ by considering $(M, F)\#(M', F')$,

$(M, F)\#(M', -F')$, *etc.* There are two possible cases: 1. These four Heegaard splittings fall into more than two equivalence classes. 2. They form at most two equivalence classes. - The first case appears more plausible than the second."

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